Financial Frictions, Liquidity Traps, and the Implementation of Monetary Policy

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Abstract

This paper studies monetary policy implementation with three different instruments: open market operations (OMO), standing facilities (SF), and lump-sum transfers (LST). We show that with financial frictions, which instrument and how we use it matters. To escape the liquidity trap, the central bank should inject more money if LST is available but reduce lending or purchase of bonds under OMO or SF. OMO and SF redistribute liquidity among borrowers and lenders and, if used aggressively, can lead the economy into the liquidity trap endogenously over time. Lending via SF is preferable to a similar OMO intervention. The relationship between the money growth rate and welfare is not simple and depends on the instrument and even the intervention size.

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1 Introduction

How do central banks implement monetary policy? The European Central Bank, the Bank of Canada, and the Reserve Bank of Australia do it mainly with standing facilities (SF), whereas the Federal Reserve System of the United States relies mainly on open market operations (OMO).[1] To achieve an inflation target, should we use OMO or SF? Which instrument is better? Given current observations with regard to interest rates and monetary policy in the United States, Japan, and many European countries, how should we apply these monetary instruments if the nominal interest is zero? The main challenge to answering these questions is not technical but, rather, the widely held belief that it does not matter how we model the specifics of monetary policy implementation.

That belief is behind a long tradition in monetary theory of assuming that central banks change the money supply by imposing lump-sum taxes/transfers (LST) (see, e.g., Tobin, 1965; Sidrauski, 1967; Grandmont and Younes, 1973; Lucas, 1980; Lagos and Wright, 2005; and Christiano et al., 2005). The thought experiment usually involves some "helicopter drops" so as to create a tax-like effect of monetary policy or an assumption that the path of lump-sum taxes changes passively to support different paths of the nominal money stock. One advantage of such an approach is, of course, simplicity. But what is the cost of taking this shortcut? What if models based on LST are not equivalent or even similar to models based on OMO or SF? What can we learn by explicitly modeling the implementation of monetary policy?

This paper studies the three implementation instruments (OMO, SF and LST) in a unified monetary framework, with transactional demand for money and an active financial market. The same belief mentioned above also leads many studies to treat the short-term nominal interest rate or the money supply as "instruments" of monetary authorities (see, e.g., Poole, 1970; Woodford, 2003; and Gali, 2008). In our model, both are endogenously determined by the intervention of monetary authorities using instruments such as SF, OMO and LST. In addition, we devote special attention to studying liquidity traps. As a starting point, the model features flexible prices and no aggregate shocks.

We show that if the financial market is perfect (i.e., without financial frictions), the three instruments are equivalent, provided that they can achieve the same money growth rate.[2] To achieve an inflation target, SF and OMO can be used either conservatively (the size of central bank intervention is constant relative to the money supply)

[1] A country using SF usually has a lending facility and a deposit facility. The US Federal Reserve System recently modified the operating procedures of its discount window facility in such a way that it now shares elements of SF.

[2] The steady-state inflation rate is the same as the money growth rate. The nominal interest rate in the financial market is given by the Fisher equation.

or aggressively (the relative size of central bank intervention grows or shrinks over time), though different ways of applying OMO or SF matter for the attainability of certain money growth rates. Whether or not the central bank reimburses its profits to the fiscal authority also matters.

If the financial market is not perfect, we have several findings. First, there are three possible types of equilibria: the unconstrained equilibrium (with slack borrowing constraint); the constrained equilibrium; and the liquidity trap equilibrium (with zero nominal interest rate in the financial market). In the latter two types of equilibria, the nominal interest rate in the financial market is depressed and lower than that implied by the Fisher equation. With each of the implementation tools, the liquidity trap equilibrium can arise if there are severe financial frictions and a shortage of government bonds, but monetary policies can also determine whether such an equilibrium is obtained.

Second, the model has qualitatively different and even opposite policy implications under LST than under the other two. With LST, a central bank can affect inflation and welfare in the liquidity trap equilibrium. It can also pull the economy out of the liquidity trap by injecting more money in the steady state. But exiting the liquidity trap does not improve welfare. In fact, the optimal policy is to remain in the liquidity trap and run the Friedman rule.[3] However, with SF or OMO, monetary policy, especially more lending or purchase of bonds, cannot marginally affect inflation or welfare in the liquidity trap equilibrium, resembling the usual notion of a liquidity trap. But the central bank can still eliminate the liquidity trap equilibrium if it reduces lending or its purchase of government bonds sufficiently, which appears to involve less injection of money. Exiting the liquidity trap improves welfare.[4]

Third, if used aggressively, monetary policy can lead the economy endogenously into the liquidity trap over time, even if the fundamentals are unchanged. Note that if SF or OMO is used in an aggressive way, a central bank needs to increase its lending or purchase of bonds over time to achieve a positive inflation target outside a liquidity trap. But, by continuing to do so, the central bank worsens the relative liquidity positions of the borrowers over time, making their borrowing constraint tighter and tighter until it binds. Welfare is then decreasing over time until the economy eventually settles in the liquidity trap equilibrium. Further injection of money in the form of central bank lending or purchase of bonds does not affect the allocation. Interestingly, the process can also be reversed (i.e., exiting the liquidity trap equilibrium) if the central bank shrinks its lending (relative to the money supply) over time.

Fourth, OMO is similar but not equivalent to SF. OMO imposes an additional "illiq-

[3]The Friedman rule refers to setting the nominal interest rate of illiquid bonds to zero, which is the same as deflation at the time preference. Williamson [2012] assumes that the central bank can use LST and get similar results.

[4]In the constrained equilibrium, LST and the other two also have very different policy implications.

uid" effect because, by purchasing and holding more bonds, the central bank reduces the amount of bonds that private agents can hold to insure against future liquidity shocks. Such an effect makes the borrowers' financial constraint tighter. However, if the central bank can issue its own bonds, then the "illiquid" effect can be reversed. Given the same relative intervention size, SF is preferred if the intervention involves the central bank lending or purchasing bonds, and OMO is preferred if the intervention involves the central bank borrowing or issuing bonds.

Fifth, the money growth rate is no longer a sufficient statistic of monetary policy. Without financial frictions, there is always a one-to-one mapping between the money growth rate and nominal interest rate. But if there are financial frictions and SF or OMO is modeled explicitly, then different monetary policies can lead to the same money growth rate and yet very different economic outcomes, even though each nominal interest rate still corresponds to a unique allocation. Depending on the policy, the appeared relationship between steady-state inflation and welfare/output can be (locally) increasing, decreasing or unrelated. The theory provides a novel rationale for not treating the money supply as an intermediate target.[5] These novel implications regarding the effects of anticipated inflation also help explain the mixed evidence on the relationship of long-run inflation and output (see Walsh [2017] and the references therein).

This paper builds on Berentsen et al. [2007] and He et al. [2015] and is related to a class of explicit models of money, liquidity, and asset exchange – "New Monetarist Economics," in the language of Williamson and Wright [2010, 2011]. Important contributions to this literature include Kiyotaki and Wright [1989], Shi [1997], and Lagos and Wright [2005]. Some recent studies focus on either OMO or SF (Berentsen and Monnet, 2008, Berentsen et al., 2014, Williamson, 2012; and Rocheteau et al., 2015).[6] These studies generate steady-state money growth by LST, whereas this paper examines how different money growth rates can be achieved by OMO or SF alone.

A useful feature of our study is our adoption of the comparative approach. Relatedly, Martin and Monnet [2011] compare SF with OMO but not with LST. We show that even though LST can be a useful modeling shortcut for SF or OMO if there are no financial frictions, it misses an important channel with the presence of financial frictions. When a central bank injects money through LST, the newly injected money reaches everyone (the usual tax-like effect) and can even generate redistributional effects that

[5]There are two main arguments against treating the money supply as a monetary policy target. First, nominal interest rates are easier to measure than the money supply. Second, if LM shocks (i.e., changes in the demand for money) are more prevalent than IS shocks (i.e., changes in the demand for goods and services), then targeting the interest rate stabilizes income better than targeting the money supply.

[6]The latter two papers also examine the illiquid effect of OMO though they do not compare OMO with SF. Berentsen and Monnet [2008] study the optimal interest-rate corridor (i.e., the interest-rate difference between the lending facility and the deposit facility), whereas we set the corridor to zero so that any difference between SF and OMO in this paper is not due to the variable corridor of SF.

favor unlucky individuals (see, e.g., Molico, 2006; and Bhattacharya et al., 2008).[7] But when a central bank lends through SF or purchases more bonds through OMO, the newly injected money worsens the relative liquidity positions of the financially constrained borrowers in the constrained equilibrium and becomes just idle cash/reserves in liquidity traps.

One of the innovations of our paper is to model the aggressive way of implementing an inflation target. In the prior literature, only Martin and Monnet [2011] study similar issues on monetary implementation, and they consider only the conservative way, which implies that the Friedman rule cannot be implemented by SF alone – in fact, if, in addition, the central bank rebates all of its profit to the fiscal authority, then it is impossible to obtain any negative money growth rate. We find that the aggressive way of implementing monetary policy not only resolves these issues, but is also more like the usual conduct of most central banks: if a central bank uses SF or OMO in the conservative way, then it needs to keep borrowing or issuing bonds to achieve a positive money growth rate, which is rarely seen in practice. Moreover, modeling the aggressive way of implementing monetary policy also leads us to the third result with financial frictions: if used aggressively, monetary policy itself can be a source of unwanted economic dynamics.

Our theory is also related to the literature on liquidity traps (see, e.g., Krugman et al., 1998; Svensson, 2000; Benhabib et al., 2002; Werning, 2011; and Eggertsson and Krugman, 2012), to studies on the interaction between financial frictions and monetary policy (see, e.g., Bernanke et al., 1999; Cúrdia and Woodford, 2010; Gertler and Karadi, 2011; and Kiyotaki and Moore, 2012), and to papers that emphasize the redistributional effect among agents with different income, wealth or access to goods or financial market, (see, e.g., Williamson, 2008; Alvarez et al., 2009; Auclert, 2016; and Kaplan et al., 2016). The value added in our article is an explicit treatment of the implementation of monetary policy, and the redistributional effect is among borrowers and lenders who have different liquidity need.

The rest of the paper is organized as follows. Section 2 starts with a brief description of the environment and then describes the explicit setup and optimal individual decisions. Section 3 discusses the model without financial frictions. Section 4 discusses the version with financial frictions. Section 5 concludes.

[7] In our model, agents' indiocyncratic shocks motivate the active financial market. If these are private information, then it is unrealistic to assume that the central bank transfers/taxes can be conditional on agents' need for liquidity.

2 Environment

2.1 Brief Description

Time is discrete. There is measure one of infinitely lived agents. Agents discount between periods with factor $\beta = 1/(1+r)$. In each period, three competitive markets open sequentially: the Financial Market (FM), the Decentralized Market (DM), and the Centralized Market (CM). In period t, agent i has periodic utility function: $-c(\ell_{it}) + \Theta_{it}u(x_{it}) + y_{it}$, where ℓ_{it}, x_{it} and y_{it} are labor supply, consumption of DM goods and consumption of CM goods. The $u(x)$ function is increasing, strictly concave and with the usual Inada condition. The $c(\ell)$ function is increasing and convex in ℓ, though special attention would be given to the case of inelastic individual labor supply (i.e., ℓ is fixed). The idiosyncratic preference shock, Θ_{it}, is independent and identically distributed across agents and over time. For simplicity, assume that the support of Θ is $\{1, \theta\}$ with α probability that $\Theta = \theta > 1$. In other words, with α probability, agents receive a preference shock that increases their marginal utility, and we call these agents high-demand consumers and the rest low-demand consumers. This setup of preference shock is a special case of the one used in Lucas (1980). Similar to Berentsen et al. (2007), the idiosyncratic shocks motivate the borrowing and lending among private agents.

Agents make their financial decisions in the FM, such as buying or selling government bonds and borrowing or lending. They also supply labor to firms in the FM. Agents then buy goods from firms in the DM, where all traders are anonymous, so that trading histories of agents are private information and firms require immediate compensation. This means that agents must pay with some assets. We assume that fiat money is the only asset that agents can use to conduct such trades. It appears to be the usual cash-in-advance constraint, but it is not because agents can acquire cash needed in the current period by borrowing and by selling their assets. Agents then receive wages and endowment of the CM goods in the CM. They can consume and trade the CM good and also trade government bonds. Although we could also allow them to borrow and lend, it would not be necessary, which will become clear later. The linear utility of y and the endowment assumption make the model tractable even with idiosyncratic shocks. As in Lagos and Wright (2005), the existence of the CM is mainly a modeling device for agents who receive idiosyncratic shocks to rebalance their asset positions.

Before agents act in the FM, four things happen sequentially: first, the idiosyncratic preference shocks are realized, motivating an active financial market; second, maturing bonds and debts are settled; third, the government (fiscal authority) issues new bonds to finance its consumption, which is fixed at G units of DM goods; last, the central bank conducts monetary policy using three alternative monetary instruments: lump-sum transfers (LST), standing facilities (SF), and open market operations (OMO). We interpret G as the part of government consumption that needs to be financed by issuing bonds. Note that the fiscal authority and the central bank operate independently in our model. The central bank buys and sells government bonds at the market rate, and as in reality, it needs to reimburse part of the operating profit to the fiscal authority. The government collects lump-sum taxes in the CM to repay its bonds that mature in the following FM. To capture the fact that the maturity of bonds is usually longer than that of short-term loans, we assume that the government bonds mature in two periods. This feature, which is missing in many previous general equilibrium models, allows the possibility of selling existing bond positions by the private agents and the central bank alike. The goal of this paper is to compare the three instruments under various assumptions about the environment.

2.2 Explicit Description

A. Private Agents

Now, we introduce the model explicitly by describing the maximization problems faced by households in each submarket. The value functions in the three markets are denoted by U, V and W. We start by introducing the CM value function in period t:

$$
\begin{aligned}
W_t\left(\tilde{m}_t,\, \hat{z}_t^{t-1},\, z_t^t,\, d_t,\, \ell_t\right) \quad &= \quad \max_{y_t,\, m_{t+1},\, z_{t+1}^t} y_t + \beta E_t\left[U_{t+1}\left(m_{t+1},\, z_{t+1}^t;\, \Theta_{t+1}\right)\right], && (1) \\
s.t. \qquad & m_{t+1} = \left(Y - y_t\right)/\phi_t + \tilde{m}_t + \ell_t w_t - d_t R_t + \hat{z}_t^{t-1}, && (2) \\
& \qquad +\hat{q}_t^t\left(z_t^t - z_{t+1}^t\right) - T_t^g, && (3) \\
& z_{t+1}^t \geq 0,\; m_{t+1} \geq 0, && (4)
\end{aligned}
$$

where the state variables in W denote the (nominal) holdings of money ($\tilde{m}_t$); government bonds issued in period $t-1$ ($\hat{z}_t^{t-1}$) and those issued in period t (z_t^t); position of one-period debt (d_t); and labor supply (ℓ_t). The superscripts denote the issuing date of the bond. Since agents supply labor in the FM but receive wages in the CM, ℓ appears as a state variable. Each unit of government bonds issued in the FM in period s would pay one unit of currency in the FM in period $s+2$.

In the budget equation, ϕ_t is the price of money in terms of the CM good, and w_t, R_t, and $\hat{q}_t^t$ are the nominal wage, gross nominal interest rate, and CM price of bonds issued in period t. Each agent is endowed with Y unit of the CM good, and her consumption of that is y_t. We assume that Y is sufficiently large so that agents have interior solutions for y.[8] T_t^g is the government lump-sum tax, which is used to pay off maturing government debt (issued in period $t-1$) in the next FM. Note that T_t^g is collected by the fiscal authority rather than by the central bank. Our definition of m_{t+1} is the net money holding after the government has collected its taxes, and all the maturing debt and bonds are settled in the next FM.[9] Since $\hat{z}_t^{t-1}$ matures in the following FM, its nominal price in the CM is simply one. One can alternatively assume that an agent holds these bonds till the following FM,and the government would simply pay $\hat{z}_t^{t-1}$ units currency, which increase m_{t+1} by exactly $\hat{z}_t^{t-1}$.

Next, we introduce agents' FM problem with realized shock Θ_t in period t:

$$
\begin{aligned}
U_t\left(m_t,\, z_t^{t-1};\, \Theta_t\right) &= \max_{\hat{m}_t, \hat{z}_t^{t-1}, z_t^t, d_t, \ell_t} -c\left(\ell_t\right) + V_t\left(\hat{m}_t,\, \hat{z}_t^{t-1},\, z_t^t,\, d_t,\, \ell_t;\, \Theta_t\right), && (5)\\
s.t. \quad & \hat{m}_t = m_t + q_t^{t-1}\left(z_t^{t-1} - \hat{z}_t^{t-1}\right) - q_t^t z_t^t + d_t + T_t^c, && (6)\\
& d\phi_t R_t \le D, && (7)
\end{aligned}
$$

where $\hat{m}_t$, $\hat{z}_t^{t-1}$, z_t^t, and d_t are the money balance, old bond, new bond and loan positions that an agent carries into the DM; q_t^t and q_t^{t-1} are the nominal prices of newly issued bonds and old bonds respectively; ℓ_t is the labor supply; R_t is the gross nominal interest rate in the loan market; and T_t^c is the lump-sum transfers given by the central bank ($T_t^c < 0$ would mean lump-sum taxes). Of course, all the asset positions must be nonnegative, while the debt position can be negative, which means lending in the FM. We further impose a borrowing constraint (7). One can interpret it in the following way: when an agent borrows in the FM, she can pledge D units of endowment goods as collateral, and in the CM, when she puts aside money to repay her due debt in the following FM, she gets D back.[10] If $D = \infty$, then there is no borrowing constraint at all.

Third, the DM problem is given by

$$
\begin{aligned}
V_t\left(\hat{m}_t,\, \hat{z}_t^{t-1},\, z_t^t,\, d_t,\, \ell_t;\, \Theta_t\right) &= \max_{x_t} \Theta_t u\left(x_t\right) + W_t\left(\tilde{m}_t,\, \hat{z}_t^{t-1},\, z_t^t,\, d_t,\, \ell_t\right), && (8)\\
s.t. \quad & \tilde{m}_t = \hat{m}_t - p_t x_t \ge 0, && (9)\\
& x_t = \hat{m}_t / p_t, && (10)
\end{aligned}
$$

where p_t is the nominal price of the DM goods. We emphasize that the cash constraint in (9) differs from the usual cash-in-advance constraint in an important regard: agents

[8] An alternative assumption would be that y could also be negative, which means that the agent is supplying labor in the CM with a linear disutility. This is a modeling device to allow agents to rebalance their asset positions.

[9] It does not matter whether the fiscal authority collects taxes in the CM or in the following FM right before they pay off their bonds.

[10] In this paper, D is fixed. In a related paper, He et al. (2016) study a monetary model in which houses can be used as collateral, and monetary policy would affect the endogenous house prices.

here do not necessarily need to prepare the money needed for consumption one period in advance; if necessary, they can borrow or sell bonds to acquire the money needed. Of course, our economy is a pure-currency economy in the sense that no Lucas and Stokey (1983) type of credit is allowed.

Firms operate competitively with linear technology, which transforms one unit of labor into one unit of DM consumption. This drives $p_t = w_t$. We assume competitive labor, goods and financial markets to focus on the comparison of monetary policy tools. See Berentsen et. al. (2012) for a monetary model with searching and match frictions in both the goods and labor markets.

B. Fiscal Authority

In the FM in period t, the fiscal authority (e.g., Department of the Treasury) receives T_t^b transfer from the central bank and issues bonds B_t^t to finance its fixed consumption of G in the DM; thus, $B_t^t = p_t G - T_t^b$. In the CM in every period t, the fiscal authority can collect lump-sum taxes $T_t^g = B_t^{t-1}$ so that it can repay the maturing debt in the following FM.

The fiscal authority in our model, though completely passive, interacts with the central bank in a realistic way: in most countries, when central banks make operating profit, they are required by law to reimburse at least part of their profit to the fiscal branch of the government; this, in turn, affects the amount of the bonds that the fiscal authority issues. For example, by lending in the financial market or purchasing government bonds the central bank would accrue an interest payment as operating profit in the following period. We assume that γ fraction of the operating profit is transferred to the government.[11]

C. Central Bank

The central bank can actively intervene in the financial market in the following three ways: (1) give lump-sum transfers, T_t^c, to each agent ($T_t^c < 0$ means taxes); (2) lend D_t^c to private agents ($D_t^c < 0$ means borrowing); or (3) purchase Z_t^t and $Z_t^{t-1} - Z_{t-1}^{t-1}$ of new and old government bonds, respectively ($Z_t^{t-1} - Z_{t-1}^{t-1} < 0$ means sales of existing bonds, while $Z_t^{t-1} \geq 0$). If the central bank chooses $T_t^c > 0$, then it would be financed by printing money. If the central bank imposes a tax instead ($T^c < 0$), then it would be different from the government taxes because the latter would always be used by the government, which makes the money supply unchanged. The requirement that $Z_t^{t-1} \geq 0$ says that the central bank can sell, at most, all of its existing holdings of old

[11]The fiscal authority has two ways to use the reimbursement. It can either use it to reduces the taxes collected required to repay maturing debt, or use it to reduce the quantity of bonds needed to finance the new consumption. We try both ways but choose to present results using the second way because government taxes are less likely to be directly affected by such reimbursement.

government bonds. For now, we do not allow the central bank to issue bonds, because this is not an option most central banks consider.[12]

Note that there is an alternative way to model standing facilities. We can let the central bank set its targeted market nominal interest rate and let the intervention size be endogenously determined. The central bank may also set a lending rate that is different from its deposit rate (see, e.g., Berentsen and Monnet 2008). As will become clear below, our approach makes it easy to compare SF and OMO. Another benefit is that whatever differences that we find between SF and OMO are not due to the extra feature of SF (i.e., the central bank is able to set the two rates differently). Furthermore, our approach leads us to find that there could be more than one way (of intervention) to achieve a policy target.

Whenever the central bank lends through standing facilities or purchases bonds, it will earn an operating profit. We assume that γ fraction of such operating profits is rebated to the fiscal authority. But if the central bank makes negative profits- for example, by borrowing via the SF- then it can simply print money to cover such costs. Thus, the transfer to the fiscal authority, T_t^b, can be written as follows:

$$T_t^b = \gamma \left\{ D_{t-1}^c \left(R_{t-1} - 1 \right) \mathbf{1} \left\{ D_{t-1}^c > 0 \right\} + Z_{t-1}^{t-1} \left(q_t^{t-1} - q_{t-1}^{t-1} \right) + Z_{t-1}^{t-2} \left(1 - q_{t-1}^{t-2} \right) \right\},$$

where the indicator function is needed because if the SF is used, then the central bank makes a profit if and only if it lends to the market. An interesting fact is that by SF, the central bank can "lose" money by borrowing from the market, but by OMO, the central bank can never lose money. By holding government bonds, the central bank always makes positive profit. Signing a repo contract does not change this result because, effectively, it just hold a smaller amount of government bonds.

D. Markets Clearing and Law of Motion

We close the model with four market clearing conditions: loans; the new and the old bonds in the FM; and goods in the DM. By Walras's law, the remaining CM good market automatically clears. The four conditions in periods t are as follows:

$$\begin{aligned} \alpha d_t(\theta) + (1-\alpha) d_t(1) &= D_t^c, &(11)\\ \alpha z_t(\theta) + (1-\alpha) z_t(1) + Z_t^t &= B_t^t, &(12)\\ \alpha \left[\hat{z}_t^{t-1}(\theta) - z_t^{t-1}(\theta) \right] + (1-\alpha) \left[\hat{z}_t^{t-1}(1) - z_t^{t-1}(1) \right] + \left(Z_t^{t-1} - Z_{t-1}^{t-1} \right) &= 0, &(13)\\ G + \alpha x_t(\theta) + (1-\alpha) x_t(1) &= \ell, &(14) \end{aligned}$$

12

- The central bank in China-that is, the People's Bank of China-does issue its own bond. It is used mainly to sterilize the increased money supply that comes with the fixed exchange rate.

where all individual choice variables, in principle, could depend on their realized idiosyncratic shocks, which are θ with probability α and 1 otherwise. The first condition says that the net position of private debt in the economy adds to the lending of the central bank. The second condition clears the new bond market in the FM. The third condition requires that the net purchase of exiting bonds adds to zero in the FM. The third condition is for DM goods.

Last, we need to describe the law of motion of the money supply. Let M_t be the money stock in circulation in the DM. Note that any flow among private agents, firms and the fiscal authority would not change the money stock in circulation. However, three possible interventions that the central bank conducts can affect the money supply: (a) from LST: T_t^c; (b) from SF: $D_t^c - D_{t-1}^c R_{t-1}$, which says that the central bank receives principal plus interest and then makes new lending in the FM; and (c) from OMO: $q_t^t Z_t^t + q_t^t \left(Z_t^{t-1} - Z_{t-1}^{t-1} \right) - Z_t^{t-2}$, the central bank takes Z_t^{t-2} units of currency out of the economy because the bonds that it holds matures; then, it injects money by (net) purchasing new and old government bonds at the market price. Of course, by SF or OMO, the central bank might earn operating profits, so we need to also take into account that it transfers T_t^b to the fiscal authority. Therefore, the law of motion of money is given by

$$M_t = M_{t-1} + T_t^c + \left(D_t^c - D_{t-1}^c R_{t-1} \right) + \left[q_t^t Z_t^t + q_t^{t-1} \left(Z_t^{t-1} - Z_{t-1}^{t-1} \right) - Z_t^{t-2} \right] + T_t^b. \tag{15}$$

There are two things to note. First, T_t^b represents a flow from the central bank to the government, which means that it is back in circulation. Second, not all money in circulation in the DM will necessarily be spent in the DM. This will be more clear later, when we consider liquidity traps. Let π_t and π_t^M denote the inflation rate and the money growth rate (in the DM) in period t-that is, $1 + \pi_t = p_t / p_{t-1}$ and $1 + \pi_t^M = M_t / M_{t-1}$. In principle, π_t can be different from π_t^M.

2.3 Efficient Allocation

In every period, the social planner would maximize $\alpha \theta u \left[x(\theta) \right] + (1 - \alpha) u \left[x(1) \right] - c(\ell)$, subject to the constraint $\ell = \alpha x(\theta) + (1 - \alpha) x(1) + G$. It is straightforward to show that the first best allocation must satisfy $\theta u_x \left[x(\theta) \right] = u_x \left[x(1) \right] = c_\ell(\ell)$. The first equality makes the marginal utility of high-demand and low-demand consumers the same, whereas the second equality makes the marginal cost equal to the benefit of the labor supply. Since $\theta > 1$, efficiency requires that $x(\theta) > x(1)$. We would also like to point to a special case in which the labor supply is perfectly inelastic and set to ℓ^c. Then, the social planner simply chooses how to allocate a fixed amount of output. The efficiently allocation would still require that $\theta u_x \left[x(\theta) \right] = u_x \left[x(1) \right]$. This special case would showcase the reallocative effects of the monetary policy.

2.4 Individual Optimal Decisions

First, consider the CM problem. Assume an interior solution of y, and we can use (2) to eliminate it in (1). Then, the first-order conditions for m_{t+1} and z_{t+1}^{t} do not depend on the realization of Θ_t and are given by

$$\phi_t = \beta \partial E_t \left(U_{t+1} \right) / \partial m_{t+1}, \quad and \quad \phi_t \hat{q}_t^t = \beta \partial E_t \left(U_{t+1} \right) / \partial z_{t+1}^t. \tag{16}$$

Similarly, the envelope conditions are

$$\partial W_t / \partial \tilde{m}_t = \phi_t, \;\; \partial W_t / \partial d_t = -\phi_t R_t, \;\; \partial W_t / \partial \ell_t = w_t \phi_t, \tag{17}$$

$$\partial W_t / \partial \hat{z}_t^{t-1} = \phi_t, \quad \partial W_t / \partial z_t^t = \phi_t \hat{q}_t^t. \tag{18}$$

Next, consider the DM problem as described by (8). Since agents carry all the state variables other than $\hat{m}_t$ directly into the CM, so the Envelope conditions of V_t with respect to the state varables are the same as that of W_t. Specifically, $\partial V_t / \partial \hat{z}_t^{t-1} = \phi_t$, $\partial V_t / \partial z_t^t = \phi_t \hat{q}_t^t$, and $\partial V_t / \partial d_t = -\phi_t R_t$. For $\hat{m}_t$, note that the liquidity constraint (9) should be binding if $R_t > 1$ because it is better to lend the extra money in the FM than to carry it in the DM. Thus, the envelope condition for V_t with respect to $\hat{m}_t$ becomes $\partial V_t / \partial \hat{m}_t = \Theta_t u_x \left(x_t \right) / p_t$.

Next, consider the problem in the FM. For ℓ, we use the envelope condition of W for ℓ and get a simplified first-order condition: $c_\ell(\ell_t) = w_t \phi_t = \omega_t$. Using the envelope conditions of V_t, we can write the first-order conditions for debt, new bonds and old bonds (i.e., d_t, z_t^t, and $\hat{z}_t^{t-1}$, respectively) are as follows

$$\Theta_t u_x \left[x_t \left(\Theta_t \right) \right] / p_t \geq \phi_t R_t, \tag{19}$$

$$q_t^t \Theta_t u_x \left[x_t \left(\Theta_t \right) \right] / p_t \geq \phi_t \hat{q}_t^t, \tag{20}$$

$$q_t^{t-1} \Theta_t u_x \left[x_t \left(\Theta_t \right) \right] / p_t \geq \phi_t, \tag{21}$$

where in each condition, equality holds when we have an interior solution, whereas inequality means a corner solution. A corner solution is possible because agents cannot hold negative bond holdings and face borrowing constraints (7). If the marginal utility in the DM is too high, it is possible for agents to borrow to the limit, to purchase zero new bonds, and to sell all the previously purchased bonds.

Condition (19) is intuitive: if one can borrow or lend without constraint, one would make the marginal benefit equal to the marginal cost, which is determined by the nominal interest rate. But why is it the nominal interest rate rather than the real interest rate that matters? The reason is that even though agents need to repay their debt only in the next period, they need to sacrifice today's (CM) consumption to put aside the money needed to pay off the debt at the beginning of the next period.[13] Because agents would access the FM before they can receive wages tomorrow, the nominal interest rate is what agents care about.

Last, from (5), we know that $\partial U_t / \partial m_t = \partial V_t / \partial \hat{m}_t = \Theta_t u_x(x_t) / p_t$, and $\partial U_t / \partial z_t^{t-1} = q_t^{t-1} \Theta_t u_x(x_t) / p_t$, so we can write the Euler equation for money holding and new bonds in the CM using (16):

$$\phi_t = \beta E_t \left[\Theta_{t+1} u_x(x_{t+1}) / p_{t+1} \right], \tag{22}$$

$$\phi_t \hat{q}_t^t = \beta E_t \left[q_{t+1}^t \Theta_{t+1} u_x(x_{t+1}) / p_{t+1} \right]. \tag{23}$$

Both equations balance the cost of acquiring liquidity and the expected benefit of liquidity. Bonds can be used for its liquidity because one can sell them for money in the FM before entering the DM.

3 Perfect Financial Market

3.1 The Equivalence Result

A financial market is perfect if no borrower is ever constrained by the borrowing constraint, (7). This will be true if either of the following assumptions is true:

$A1$. $\alpha = 0$.

$A2$. $\alpha \in (0, 1)$ and $D = \infty$.

If $A1$ is true, then agents are always homogeneous, so there is no reason for them to borrow from and lend to each other in the FM. The financial market is perfect in the trivial sense. If $A2$ is true, then there is an active motive for high-demand consumers to borrow from low-demand consumers, and they can borrow as much as they want. In both cases, the first-order conditions, (19), (20) and (21), hold with equality for every agent.[14] The bond prices are, thus, given by

$$q_t^{t-1} = 1/R_t, \textit{ and } q_t^t = \hat{q}_t^t / R_t = E_t \left[1/\left(R_t R_{t+1}\right) \right], \tag{24}$$

where we have used the fact that $\hat{q}_t^t = E_t\left(q_{t+1}^t\right)$, which is obtained by combining the

[13] Camera et. al. (2005) find a similar result, though they directly assume that agents pay off their debt in the same period. For concreteness, suppose that a period is a day. The FM opens from 8am to 9am. Then, the DM opens from 9am to 5pm, and agents work and consume the DM goods. Finally, the CM opens from 5pm to 11pm, when agents receive wages and trade CM goods and bonds before going to bed at 11pm. To repay the debt in the FM tomorrow means that agents need to sacrifice some CM consumption today.

[14] If $A1$ is true, we can still price the loan, even though the representative agents have zero debt position.

two Euler equations, (22), and (23).

Below, we characterize the equilibrium. First, since equation (19) holds with equality for both type of agents, $\theta u_x [x_t(\theta)]$ is equal to $u_x [x_t(1)]$. Combining (19) with (22), we have $\phi_t = \beta E_t (R_{t+1}\phi_{t+1})$. With a perfect financial market, all of the money supply is used to purchase the DM goods. So, we must have $p_t \ell_t = M_t$ for any t. From (19), we have $u_x [x_t(1)] = p_t \phi_t R_t$ for any t. Since $\phi_t = u_x [x_t(1)] / p_t R_t$, we have the following form of the Euler equation and the market clearing condition in the DM

$$u_x [x_t(1)] / R_t = \beta E_t \left\{ \frac{u_x [x_{t+1}(1)]}{1+\pi_{t+1}} \right\}, \tag{25}$$

$$\ell_t - G = \alpha u_x^{-1} \{u_x [x_t(1)] / \theta\} + (1-\alpha) x_t(1), \tag{26}$$

Let r_t be the real interest rate (in the FM) defined by the usual Fisher equation

$$1 + r_t = R_t E_t \left(\frac{1}{1+\pi_{t+1}} \right). \tag{27}$$

If the labor supply is inelastic, then the allocation is always efficient (i.e., with any monetary policy under any policy tools) because in this case, efficiency requires only that the marginal utility is equalized across agents. Furthermore, ℓ_t is constant, so (26) implies a unique (and constant) solution of $x_t(1)$ and, thus, $x_t(\theta)$. According to (25), we have $R_t E_t [1/(1+\pi_{t+1})] = 1/\beta$. Because $p_{t+1} = M_{t+1}/\ell$, we have $\pi_t = \pi_t^M$, and the gross nominal interest rate depends only on the expected money supply in the next period. We also have $1 + r_t = 1/\beta$. The real interest rate implied by the Fisher equation is constant and is equal to that implied by the discount factor.

Next, consider the elastic labor case. We still have $p_t \ell_t = M_t$, but now ℓ_t is no longer fixed. Therefore $x_t(1)$ and $x_t(\theta)$ can change over time. Using $c_\ell(\ell_t) = \omega_t = p_t \phi_t$, we can write $u_x [x_t(1)] = c_\ell(\ell_t) R_t$. Then, the market clearing condition, (26), can be rewritten as

$$\alpha u_x^{-1} [c_\ell(\ell_t) R_t / \theta] + (1-\alpha) u_x^{-1} [c_\ell(\ell_t) R_t] = \ell_t - G, \tag{28}$$

which says that the gross nominal interest rate, R_t, can uniquely pin down the labor supply, ℓ_t, and, thus, $x_t(1)$ in period t. The Euler equation (25) can be written as

$$u_x [x_t(1)] \frac{\ell_t}{M_t} \frac{1}{R_t} = \beta E_t \left\{ u_x [x_{t+1}(1)] \frac{\ell_{t+1}}{M_{t+1}} \right\}, \tag{29}$$

where (given M_t) the left-hand side is a function of R_t and the right-hand side is a function of R_{t+1} and M_{t+1}. Thus, equations (28) and (29) define a dynamic system. Given the sequence of M_{t+1}, we can calculate the implied R_t and ℓ_t sequence. Note that the relationship $r_t = r$ does not necessarily hold out of steady state.

In the steady-state equilibrium in which ℓ_t and R_t are endogenous but constant over time, we must have a constant growth rate of M_t, and, thus, $\pi = \pi^M$. Then, $r_t = r = 1/\beta - 1$. We can plug the R into (28) to obtain the steady state ℓ, and it is obvious that ℓ is decreasing in R. Euler equation (29) simply implies the usual Fisher equation. To show that the Friedman rule (i.e., setting $1 + \pi = 1/\beta$) achieves the first best, we can write the steady-state version of (19) as

$$c_\ell(\ell) \frac{1+\pi}{\beta} = u_x[x(1)]. \tag{30}$$

It is clear that the expected inflation drives a wedge between the societal marginal cost and the marginal benefit of DM consumption, as in standard monetary theory. If the labor supply is inelastic, then expected inflation pushes down the real wages of workers but leaves the total output unchanged.

So far, we have shown how to characterize the equilibrium. Notice that, at this point, as long as we know the dynamics of the money supply, we do not need to know the specifics of the monetary tools being used. The above results are summarized in the following proposition:

Proposition 1. *If the financial market is perfect (i.e., either $A1$ or $A2$ is satisfied), then (a) the marginal utility is equalized across agents; (b)* ***the equilibrium allocations depend only on the growth rate of money supply and not on the specific monetary tools that are used;*** *(c) there exists a unique steady-state equilibrium for any constant growth rate of the money supply; (d) the gross nominal interest rate is $R = \left(1 + \pi^M\right)/\beta$ in the steady state; (e) if the labor supply is inelastic, then the allocation is always efficient, and $r_t = r = 1/\beta - 1$ even out of the steady state; and (f) if the labor supply is elastic, the steady-state labor supply and consumption is decreasing in the growth rate of the money supply; the first best allocation is achieved if the growth rate of the money supply is set to $\beta - 1$; and $r_t = r$ might not hold out of the steady state.*

Part (b) is our equivalence result. Despite the wide usage of LST to model the implementation of monetary policy, there are questions regarding whether deflation and, especially, the Friedman rule are incentive-feasible under LST (see, e.g., Andolfatto 2013). We believe that our environment is general enough that other models with transaction demand of money would share the same insight: that models based on LST would be the same if we simply change the policy tool to SF or OMO, provided that there is a perfect financial market and that these policy tools can achieve the same money growth rate and inflation rate. If so, we can simply use LST as a simplifying model assumption.

However, Proposition 1 does not say whether a central bank can use SF or OMO alone to achieve a money growth rate that is theoretically possible if LST is used to implement monetary policy. This question is rarely studied. An exception is Martin and Monnet [2011], who claim that Friedman rule cannot be achieved with SF alone. If so, then a model with LST in which the Friedman rule is optimal cannot be applied to a country whose central bank uses SF to implement monetary policy. Below, we explore how different monetary policy can be implemented by different tools.

3.2 Implementing Monetary Policies

To compare different monetary policy tools, we study them in isolation. The emphasis would be put on implementing different steady-state inflation rates. Note, again, that, in general, $\pi_t^M \neq \pi_t$, but whenever we discuss the steady state, we must have$\pi^M = \pi$.

3.2.1 LST Only

Suppose that the central bank changes the money supply through LST only. Let $T_{t+1}^c / M_t = \tau_t$. From the law of motion of M_t, (15), we have $\pi_{t+1}^M = \tau_t$. If $\tau_t = \tau$ for any t, then it is obvious that we have constant growth of the money supply and $\pi = \tau$. We have inflation (deflation) if the central bank imposes lump-sum transfers (taxes). Either steady inflation or deflation can be achieved if τ is held constant. The Friedman rule can be achieved by setting $\tau = \beta - 1$ so that $\pi = \beta - 1$ and $R = 1$. These results are standard in monetary economics and serve as a benchmark for comparisons below.

3.2.2 SF Only

Next, consider SF. When the central bank lends to the market, $D_t^c > 0$, it earns a profit of $D_t^c(R_t - 1)$ in the next FM and transfers $T_{t+1}^b = \gamma D_t^c(R_t - 1)$ to the fiscal authority. Let $\hat{M}_t$ be the after-settlement money stock in the financial market (i.e., after settlement but before the central bank intervenes); then, $\hat{M}_t = M_{t-1} + T_t^b - D_{t-1}^c R_{t-1}$. Assume that $D_t^c = \delta_t \hat{M}_t$, where δ_t represents the relative size of the central bank's intervention. Of course, positive (negative) δ_t means that the central bank is lending (borrowing). Notice, also, that $\hat{M}_t = \hat{M}_{t-1} - D_{t-1}^c(R_{t-1} - 1)\left[1 - \gamma \mathbf{1}(\delta_{t-1} > 0)\right]$, where $\mathbf{1}(\delta_{t-1} > 0)$ is an indicator function that equals one if $\delta_{t-1} > 0$ and zero otherwise. Note that the after-settlement money stock decreases or increases only because the central bank keeps or loses profit. We have an expression for the growth rate of the money supply in circulation (i.e., in the DM):

$$1 + \pi_t^M = \frac{M_t}{M_{t-1}} = \frac{(1 + \delta_t)}{(1 + \delta_{t-1})} \left[1 - \delta_{t-1}\left(R_{t-1} - 1\right)\left[1 - \gamma \mathbf{1}(\delta_{t-1} > 0)\right]\right]. \tag{31}$$

Conservative Intervention Next, consider the case in which $\delta_t = \delta$ for any t. Because the size of the central bank intervention is constant relative to $\hat{M}_t$, we call it the "conservative intervention." The steady-state version of (31) is given by

$$1+\pi^{M}=1-\delta\,(R-1)\,[1-\gamma\mathbf{1}(\delta>0)]. \tag{32}$$

If we further impose $\beta R = 1+\pi^{M}$, which is true with the perfect financial market, then we have

$$R=\frac{1+\delta[1-\gamma\mathbf{1}(\delta>0)]}{\beta+\delta[1-\gamma\mathbf{1}(\delta>0)]}, \tag{33}$$

where $\delta > -\beta$.[15] Note that such an expression might not be true if the high-demand consumers are constrained in the FM. If $\delta > 0$, then γ matters for the nominal interest rate, as well. The property of steady-state inflation is summarized in the following Lemma:

Lemma 1. *With a perfect financial market, suppose that the central bank uses a standing facility only, and the relative size of central bank lending, $\delta_t > -\beta$, is constant over time. Then, we must have (i) if $\delta = 0$ or $\delta > 0$ and $\gamma = 1$, then $\pi = 0$; (ii) if $\delta > 0$ and $\gamma \in (0,1)$, then $\pi < 0$; (iii) if $\delta < 0$, then $\pi > 0$; (iv) if $\gamma \in (0,1)$ or if $\gamma = 1$ and $\delta < 0$, then $\partial\pi/\partial\delta < 0$; and (v) the Friedman rule cannot be achieved with finite δ.*

The property of R is similar because of the steady-state Fisher equation. For results (i)~(iii), notice that if δ is constant, the only way for the central bank to increase (decrease) the money supply over time is to keep borrowing (lending) in the FM so that the interest payment paid (collected) by the central bank goes into (out of) circulation. It is also interesting to note that if $\gamma = 1$, then we cannot implement any deflation rate.

The result (v) that the Friedman rule cannot be achieved by any finite δ can be seen from (32). Suppose that it can be achieved so that $R = 1$ and $\pi = \beta - 1$; then, the RHS of (32) equals one, whereas the LHS is β . This is a contradiction. Technically, we need to set $\delta \to +\infty$ to approach the Friedman rule. It is clear from (32) that the lower the nominal interest rate, the larger the relative intervention that is needed to further lower the inflation.

But is there still a way that we can achieve the Friedman rule with SF alone? After all, so many models based on LST claim that the Friedman rule is the optimal policy. Another intriguing result is that inflation is always zero if $\delta > 0$ and $\gamma = 1$, and to achieve positive inflation, we need $\delta < 0$. These results appear to be at odds with the usual observation that when the central bank lends more to the market, inflation increases. The general impression is that central bank lending is responsible for inflation. Below, we explore these possibilities.

[15] The technical condition $\delta > -\beta$ is required because, otherwise, there is no $R \geq 1$ that can satisfy the Fisher equation and (32) at the same time. If $\delta < -\beta$, then only $R < 1$-i.e., the negative net nominal interest rate-can satisfy the conditions. But that would mean arbitrage opportunities, just as $\tau < \beta - 1$ would mean with LST only.

Aggressive Intervention It is natural to think that a stationary equilibrium requires the relative size of intervention, δ_t, to be constant over time (see Martin and Monnet 2011). However, once we remove this restriction, we find that with SF alone, more lending can indeed lead to more inflation, and the Friedman rule can also be achieved. We offer the following lemma.

Lemma 2. *With a perfect financial market and standing facility only, (i) the Friedman rule can be achieved by setting $\delta_t = \beta\delta_{t-1} + \beta - 1$, so δ_t converges to -1 overtime; and (ii) even with $\gamma = 1$, any constant inflation rate π can be achieved if $\delta_t = \pi + (1+\pi)\,\delta_{t-1}$.*

The first part of the Lemma is from (31): by setting $R_{t-1} = 1$ and $1 + \pi_t^M = \beta$. The second part of the Lemma is from (31): if $\gamma = 1$ and we want to achieve some constant inflation rate π, then we just need to set $\delta_t = \pi^M + \left(1 + \pi^M\right)\delta_{t-1}$. Of course, this way of achieving a level of inflation means that the balance sheet of the central bank becomes larger and larger (relative to the size of GDP or the money supply). This way of conducting policy is closer to what central banks are doing in reality – we call it aggressive intervention (as oppose to the conservative option, which keeps δ constant). Conservative interventions rely entirely on the profit (can be negative) of interventions to change the money supply. Aggressive interventions rely mainly on the intervention itself to change the money supply, which has some unintended consequences if there is financial friction. We will return to these issues in Section 4.

3.2.3 OMO Only

Now we study OMO. To simplify the analysis, we consider two options for the central bank. The first option is to buy only old government bonds. The central bank buys $Z_t^{t-1} > 0$ units of old bonds and sets $Z_t^t = 0$. Similar to our discussion of SF, it is useful to denote $\mu_{ot} = q_t^{t-1} Z_t^{t-1} / \hat{M}_t$ as the relative size of intervention, where $\hat{M}_t = M_{t-1} - Z_{t-1}^{t-2} + \gamma\left(1 - q_{t-1}^{t-2}\right) Z_{t-1}^{t-2}$ is the after-settlement money stock in the FM. The second option is to buy and hold only newly issued bonds.[16]

With either option, since M_t is different from $\hat{M}_t$ only because of the central bank intervention, $M_t = (1 + \mu_{it})\,\hat{M}_t$, where i can be o or n, indicating buying and holding only old bonds or buying and holding only new bonds. We have an expression for the growth rate of the money supply in circulation:

$$1 + \pi_t^M = \frac{M_t}{M_{t-1}} = \frac{(1 + \mu_{it})}{(1 + \mu_{it-1})}\left[1 - \mu_{it-1}\left(R_{t-1} - 1\right)(1 - \gamma)\right]. \tag{34}$$

This expression is similar to (31), except tha tnow we replace δ_t with μ_{it} and get rid of

[16]The second option is to set $Z_t^t > 0$ and $Z_{t+1}^t = 0$. Let $\mu_{nt} = q_t^t Z_t^t / \hat{M}_t$ be the relative size of intervention, where $\hat{M}_t = M_{t-1} - Z_{t-1}^{t-1} + \gamma\left(q_t^{t-1} - q_{t-1}^{t-1}\right) Z_{t-1}^{t-1}$.

the indicator function because μ_{it} is always nonnegative. If we allow the central bank to issue its own bonds, then the two conditions will be almost the same.

Conservative Intervention In a steady-state equilibrium where μ_{it} is constant over time, we have

$$1 + \pi^M = 1 - \mu_i (R - 1)(1 - \gamma), \tag{35}$$

which is similar to (32). Since μ_i is naturally nonnegative, $\pi \leq 0$. Given our discussion of SF, this should not be surprising. Using the steady-state Fisher equation, we can write $R = [1 + \mu_i (1 - \gamma)] / [\beta + \mu_i (1 - \gamma)]$, which is similar to (33). It does not matter whether the central bank deals with new or old bonds. We will revisit this property when we discuss financial frictions.

Lemma 3. *With a perfect financial market, suppose that the central bank uses open market operations only, and the relative size of central bank intervention, $\mu_{it} \geq 0$, is constant over time. Then, we must have (i) if $\mu_i = 0$ or $\mu_i > 0$ and $\gamma = 1$, then $\pi = 0$; (ii) if $\mu_i > 0$ and $\gamma \in (0, 1)$, then $\pi < 0$; (iv) if $\gamma \in (0, 1)$, then $\partial \pi / \partial \mu_i < 0$; and (v) the Friedman rule cannot be achieved with finite μ_i.*

These results are similar to Lemma 1, except that now μ_i cannot be negative. Can the central bank use a repo contract to effectively borrow money from the market in order to gradually increase the money supply? The answer is no. At least not in the steady state. Suppose that the central bank purchases Z_t^t at q_t^t and then signs a repo contract: it sells these bonds at some price now and purchases the bonds next period at some other price. If both prices are market prices, then these transactions simply cancel out the initial purchase.[17]

Aggressive Intervention What if we allow μ_{it} to vary over time? Consider the new-bond option taken by the central bank. First, suppose that the central bank would like to implement a constant money growth rate $\pi_t^M = \tilde{\pi}$ in the steady state, which implies a constant gross nominal interest $\tilde{R} = (1 + \tilde{\pi}) / \beta$. If $\tilde{\pi} > 0$, then (34) implies that the ratio $(1 + \mu_{it}) / (1 + \mu_{it-1})$ is increasing over time. In such an accelerating fashion, $\lim_{t \to \infty} \mu_{it} = \infty$. This is a problem for OMO since there might not be enough government bonds. To see this, note that $\mu_{n,t} = q_t^t Z_t^t / \hat{M}_t \leq q_t^t B_t^t / \hat{M}_t$ since there are, at most, B_t^t units of nominal bonds in period t. Note, also, notice that $q_t^t B_t^t = G p_t - T_t^b$. If $\gamma = 0$, then after some algebra, we arrive at the following constraint:

$$\mu_{nt} (1 + \mu_{nt}) \leq \frac{G}{\ell}. \tag{36}$$

[17]This is the case unless we have some deep theory about why the two prices involved in the repo contract should be different than the market prices. A reverse repo contract means temporarily lending, so it cannot help the central bank either.

If $\gamma > 0$, then the inequality would be even tighter because T_t^b would be positive, so the fiscal authority would issue fewer bonds to finance its expenditure.[18] The central bank can start buying old bonds (or, in other words, not sell the old bonds), as well. But that option would be exhausted eventually, too. Of course, the fiscal authority can accommodate the OMO by rolling over larger and larger amounts of government debt. But that is not entirely innocuous given other the consequences of higher government debt.

Next, suppose that the central bank's inflation target is $\tilde{\pi} < 0$. If $\tilde{\pi}$ is lower than $-\mu_{it-1}\left[\left(1+\tilde{\pi}\right)/\beta - 1\right]\left(1-\gamma\right)$, then the ratio $\left(1+\mu_{it}\right)/\left(1+\mu_{it-1}\right)$ is decreasing over time. If, further, $\gamma = 1$, then according to (34), any negative inflation rate requires μ_{it} to decrease over time and eventually become negative. This is not feasible in our current setting unless the central bank can issue its own bonds.[19] The Friedman rule, in particular, also cannot be implemented.[20] If we allow the central bank to issue its own bonds, so that μ_{it} can be negative, then deflation and the Friedman rule can be achieved as with the SF.[21]

It is useful to summarize the findings in this subsection. With no financial frictions, if the central bank uses either SF or OMO to implement monetary policies, then there are two options to achieve a steady-state monetary equilibrium: the conservative option (by holding the relative intervention size constant) or the aggressive option (by increasing the size of the intervention over time). The Friedman rule can be achieved with the aggressive option (with OMO, the central bank needs to issue its own bond) but not with the conservative option. Without the ability to issue its own bonds, the central bank can obtain positive inflation only with the aggressive option – i.e., buying up more and more government bonds, which requires the fiscal authority to increase the bond supply over time.

4 With Financial Frictions

In this section, we consider the case in which $\alpha \in (0,1)$ and $D < \infty$-i.e., there is always an active financial market with financial frictions. Below, we first discuss some of the

[18] If $\gamma > 0$, then the exact expression of T_t^b/p_t is

$$T_t^b/p_t \quad = \quad \gamma\ell\frac{\mu_{n,t-1}}{1+\mu_{n,t}}\left(R_{t-1}-1\right)/\left[1-\mu_{it-1}\left(R_{t-1}-1\right)\left(1-\gamma\right)\right]$$

[19] The People's Bank of China regularly issues its own bonds to manage the liquidity in the financial market.

[20] For the Friedman rule, technically, we need to set $\mu_{it} = \beta\mu_{it-1} + \beta - 1$, which means that μ_{it} needs to decrease over time and converge to -1.

[21] Martin and Monnet (2011) let the central bank initially acquire a large sum of long-term government bonds, and gradually reduce its bond holdings over time. This is similar to the central bank issuing more and more bonds but with the possibility of running out of bond position in the long run.

general conditions that hold true under all of the three policy tools in Section 4.1. Then, we examine each policy tool in isolation.

4.1 General Conditions

Because in the FM, low-demand consumers (i.e., agents with $\Theta = 1$) tend to be lenders, we start by assuming that the borrowing constraint (7) is not binding for them, which is always true, but we defer the discussion of this issue below. Their first-order conditions of loans and, new and old bonds-i.e., (19), (20) and (21)-thus always hold with equality. Bond prices still satisfy condition (24). The Euler equation for money demand and the market clearing condition can be written as

$$\begin{aligned} \phi_{t-1} &= \beta E_{t-1}\left\{\alpha\theta u_x\left[x_t\left(\theta\right)\right]/p_t + \left(1-\alpha\right)\phi_t R_t\right\}, & (37)\\ \ell_t &= \alpha x_t\left(\theta\right) + \left(1-\alpha\right)u_x^{-1}\left[\omega_t R_t\right] + G, & (38)\end{aligned}$$

where ℓ_t is $c_\ell^{-1}\left(\omega_t\right)$ or constant depending on whether individual labor supply is elastic or inelastic. Notice that $\theta u_x\left[x_t\left(\theta\right)\right]/p_t \geq \phi_t R_t = u_x\left[x\left(1\right)\right]/p_t$, and the inequality holds when high-demand consumers are strictly constrained in the FM. In that case, condition (19) must hold for them with inequality (i.e., for agents with $\Theta = \theta$). Similarly, conditions (20) and (21) must also hold for them with inequality.

Intuitively, high-demand consumers would like to avoid hitting the borrowing constraint by selling some of their old bonds and by buying fewer new bonds. Only after they have exhausted their resources-that is, after selling all of their existing bonds and reducing new bond purchases to zero-would they face binding borrowing constraints.[22] Since $x_t\left(\theta\right)p_t = \hat{m}_t$, we can write down the expression of $x\left(\theta\right)$ using (6) after some algebra:

$$x_t\left(\theta\right)p_t = m_t + \tau M_{t-1} + q_t^{t-1}z_t^{t-1} + D/\phi_t R_t, \tag{39}$$

where τM_{t-1} is the LST imposed by the central bank.

Steady-state Equilibrium Next, we consider the steady-state equilibrium with constant τ, δ, and μ. The market clearing condition is straightforward from equation (38), whereas it is useful to write down the steady-state version of the Euler equation,

$$\frac{1+\pi}{\beta} = \alpha\theta u_x\left[x\left(\theta\right)\right]/\omega + \left(1-\alpha\right)R, \tag{40}$$

[22]Note that if $\alpha = 0$-that is, if there are no idiosyncratic shocks-then there is no trade of old bonds, or, in other words, the price of bonds would just make every agent indifferent between selling and purchasing bonds.

which is different from (30), because now $\theta u_x [x(\theta)] \geq u_x [x(1)] = \omega R$. We immediately have the the following result:

Proposition 2. *In the steady state, if financial friction is strictly binding for high-demand consumers, then the nominal interest rate in the FM satisfies $R < (1+\pi)/\beta$, and the real interest rate in the FM is lower than $r = 1/\beta - 1$.*

This proposition says that the nominal interest rate R in the FM is depressed by financial frictions: high-demand consumers cannot borrow as much as they would like to. Otherwise, R would have gone up to $\theta u_x [x(\theta)] / \omega$, and we would have $R = (1+\pi)/\beta$. The real interest rate in the FM, $R/(1+\pi) - 1$, is also depressed by financial frictions, as it is smaller than $1/\beta - 1$. Due to condition (24), the nominal interest rate of the short-term financial market is the same as the implied nominal yield of the liquid government bond. Thus, we have similar results for bond returns:

Corollary 1. *In the steady state, if financial frictions are strictly binding for high-demand consumers, then there exists a liquidity premium for the government bond both in the CM and in the FM.*

For the proof, see the Appendix. Why do agents carry these "overpriced" government bonds across periods? They do so because these bonds provide liquidity. In Williamson (2012) and Rocheteau et al. (2014), the government bond itself can be used as a medium of exchange and, thus, may have a liquidity premium. Here, bonds cannot be used to buy goods but are, nevertheless, "liquid" because agents can sell them when they need money. The liquidity premium for bonds exists only if financial frictions are binding with positive probability.

Critical Values of Borrowing Constraint Though the borrowing limit, D, is exogenous, whether it is binding is endogenous. Given the policy tool and the size of intervention, we can define two cutoffs D_u and D_ℓ, with $D_u \geq D_\ell$ such that (a) none of the agents is financially constrained if and only if $D \geq D_u$; and (b) no idle cash is carried into DM if and only if $D \geq D_\ell$. The idle cash is not used in DM for sure. It is different from the unused money balances in Lagos and Wright (2005), where every unit of money is used with positive probability. Given these two cutoffs, we have three possible types of steady-state equilibria: the unconstrained equilibrium (i.e., $D \geq D_u$); the constrained equilibrium (i.e., $D_u > D > D_\ell$); and the liquidity trap equilibrium (i.e., $D \leq D_\ell$). In the first two types of equilibria, $R > 1$, the loan market clears and there is no idle cash in the DM, so that $M_t = p_t \ell = w_t \ell = \omega \ell / \phi_t$. In the liquidity trap equilibrium, $R = 1$ and the loan market does not clear. Williamson (2012) points out that the liquidity trap equilibrium does not exactly correspond to the Friedman rule, which should require that $\pi = \beta - 1$. Suppose that agents can trade a bond that is not liquid in the financial market; then, only when $\pi = \beta - 1$ would the nominal interest of such a bond be zero.

In the unconstrained equilibrium, we have $\theta u_x [x(\theta)] = u_x [x(1)] = \omega R$, and the goods market clearing condition becomes

$$\alpha u_x^{-1} (\bar{\omega}/\theta) + (1-\alpha) u_x^{-1} (\bar{\omega}) + G = \ell, \tag{41}$$

where $\bar{\omega} = \omega R$. Apparently, there is a unique solution of $\bar{\omega}$. We can then use equation (39) with $D = D_u$ and condition $\theta u_x [x(\theta)] = \bar{\omega}$ to pin down the value of D_u.

In the liquidity trap equilibrium, let the real wage be $\underline{\omega}$; then, $u_x [x(1)] = \underline{\omega}$ because $R = 1$. The goods market clearing now requires that

$$\alpha u_x^{-1} \left[\left(\frac{1+\pi}{\beta} - 1 + \alpha \right) \frac{\underline{\omega}}{\alpha\theta} \right] + (1-\alpha) u_x^{-1} (\underline{\omega}) = \ell - G, \tag{42}$$

where we have applied the Euler equation in this case: $\alpha\theta u_x [x(\theta)] = \left(\frac{1+\pi}{\beta} - 1 + \alpha \right) \underline{\omega}$. Given the inflation rate, equation (42) uniquely pins down the value of $\underline{\omega}$. We can then use equation (39) with $D = D_\ell$ to determine D_ℓ.

If D_u is negative, then it means that agents are not financially constrained for any nonnegative D. If D_ℓ is negative, then it means that we will never have the liquidity trap equilibrium. However, because the expressions of D_u and D_ℓ might depend on the specific policy tools, we proceed by examining each of the policy tools in isolation, as in the previous section.

Because the tax-like effect of monetary policy is well understood, in the following analysis, we assume that labor supply is inelastic to focus on the redistributional effects.

4.2 LST Only

We now look at the case of LST. In principle, the two cutoffs of the borrowing limit, $D_u^\tau(\tau)$ and $D_\ell^\tau(\tau)$, are functions of τ, where the superscript τ means that LST is used for implementing monetary policy. Since the central bank does not buy any government bonds, $z_t^{t-1} = B_{t-1}^{t-1} = Gp_{t-1}/q_{t-1}^{t-1}$. Thus, we have $q_t^{t-1} z_t^{t-1} / p_t = RG/(1+\tau)$ in the steady state.[23] We can write the steady-state version of equation (39) as

$$x(\theta) = M/p + GR/(1+\tau) + D/\omega R, \tag{43}$$

where $\ell = M/p$ if there is no idle cash in the DM. Using the above equation, $x(1) = u_x^{-1}(\omega R)$ and market clearing conditions, we have the following expressions of the

[23]If there is of the bonds (i.e., $R < (1+\pi)/\beta$), then the bonds would be worth less than G/β units of DM goods. This is another way to see that the bonds are "overpriced" above its fundamental.

two cutoffs of the borrowing constraint:

$$D_u^\tau(\tau) = D_u^\tau = \bar{\omega}\left\{\frac{1-\alpha}{\alpha}\left[\ell - u_x^{-1}(\bar{\omega})\right] - \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)G\right\}, \tag{44}$$

$$D_\ell^\tau(\tau) = \underline{\omega}\left\{\frac{1-\alpha}{\alpha}\left[\ell - u_x^{-1}(\underline{\omega})\right] - \left(\frac{1}{\alpha}+\frac{1}{1+\tau}\right)G\right\}, \tag{45}$$

where $\bar{\omega}$ and $\underline{\omega}$ are defined in equations (41) and (42), respectively.

There are three observations. First, if $1+\tau=\beta$, then $D_\ell^\tau(\tau) = D_u^\tau(\tau)$. Second, if $G=0$, then both $D_u^\tau(\tau)$ and $D_\ell^\tau(\tau)$ are positive. So all three types of equilibria are possible. However, if G is large enough, then both critical values can be negative. For example, if $D_u^\tau < 0$, then agents are never constrained. The government bonds alone provide enough liquidity insurance for the private agents. Third, τ affects $D_\ell^\tau(\tau)$ but not $D_u^\tau(\tau)$.[24] From (42) and (45), we know that $D_\ell^\tau(\tau)$ is decreasing in τ. Given $D < D_u^\tau(\tau)$, it is likely to be smaller than $D_\ell^\tau(\tau)$ when τ is small and greater than $D_\ell^\tau(\tau)$ when τ is large. Panel (a) of Figure 1 plots an example of D, $D_u^\tau(\tau)$, and $D_\ell^\tau(\tau)$.

Why does the liquidity trap equilibrium (i.e., $D < D_\ell^\tau(\tau)$) tend to happen when inflation is low? It does so because lower inflation increases real money balances (in terms of AD goods). If inflation is low enough, then the borrowing limit is not enough for borrowers to borrow all the extra real money balances from lenders. So lenders carry idle cash into the DM. The fixed borrowing limit assumed in this paper is not crucial for this argument. Similar results hold in He et al. (2015), in which the borrowing constraint depends on the endogenous housing price.

After characterizing the boundary conditions, we can think about policy implications under the three regimes. First, if $D > D_u^\tau(\tau)$, both types of agents are not constrained, then we must have $R = (1+\tau)/\beta$ and $\theta u_x[x(\theta)] = u_x[x(1)] = \bar{\omega} = \omega R$. An increase in τ raises R, lowers ω but does not affected $x(\theta)$ or $x(1)$.

Second, if $D_u^\tau(\tau) > D > D_\ell^\tau(\tau)$, then we are in the constrained equilibrium. The equilibrium conditions are steady-state versions of (37) and (38), with $\pi=\tau$, $p\phi = \omega$, and $x(\theta)$ is given by equation (43). So we have a two-equation system with two unknowns: ω and R. Interestingly, an increase in τ raises R, lowers ω, but does not affect $x(\theta)$ or $x(1)$. However, since $D_\ell^\tau(\tau)$ is decreasing in τ, given $D < D_u^\tau(\tau)$,there exists a $\tilde{\tau}$ such that whenever $\tau \le \tilde{\tau}$, we have $D \le D_\ell^\tau(\tau)$.

Third, if $D < D_\ell^\tau(\tau)$, then we are in the liquidity trap equilibrium and there is idle cash in the DM. We can eliminate the $x(\theta)$ in the Euler equation (37) using the market clearing condition of (38):

$$\frac{1+\pi}{\beta} = \alpha\theta u_x\left\{\frac{1}{\alpha}\left[\ell - G - (1-\alpha)u_x^{-1}(\omega)\right]\right\}/\omega + (1-\alpha), \tag{46}$$

[24] $D_u^\tau(\tau)$ is not affected by τ only if the labor supply is inelastic.

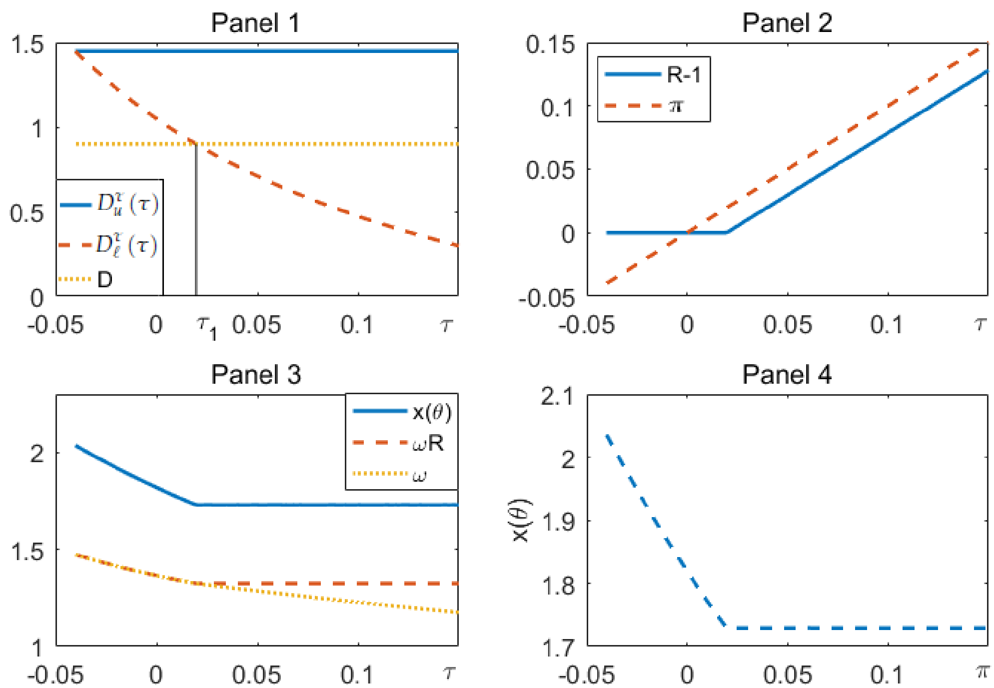

Figure 1: Effects of Lump-sum Transfers/Taxes

where $\pi = \tau$. Because $R = 1$, we have one equation and one unknown, ω. An increase in τ now lowers ω, raises $x(1)$ and reduces $x(\theta)$. Note that these results are true even if $G = 0$. If $\tau = \beta - 1$, then $\theta u_x[x(\theta)] = u_x[x(1)]$. So the Friedman rule can be achieved, and it delivers the efficient allocation.

Note that these differ from the usual result that inflation decreases market activity because, here, the labor supply is fixed.[25] In the liquidity trap equilibrium, lower inflation induced by LST has three effects on the steady states according to (43): first, agents' government bond holdings are worth more DM goods (i.e., $GR/(1+\tau)$ is decreasing in τ when $R = 1$). But this is not a crucial effect because the results would be the same even if $G = 0$ according to (46). Second, the money supply in the DM is worth more than the DM goods. Note that this is not true when there is no idle cash in the DM: in that case, the money supply in the DM is always worth exactly the entire amounts of the DM goods. Lower inflation always raises the real money balances (in terms of AD goods). In the liquidity trap equilibrium, it implies now that what borrowers can borrow, which is fixed in terms of AD goods, is a smaller fraction of lenders' money.[26] More idle cash means that a smaller fraction of the money supply is worth all the DM goods. This is good for borrowers because they can use their after-intervention money balances, M, to afford more DM goods. Third, agents' borrowing

[25] If we also assume elastic labor, then the labor supply would be decreasing in inflation, as well.

[26] Again, in He et al. (2015), there is similar result even though the borrowing limit is not exogenous and depends endogenously on housing prices. They do not have government bonds.

constraint is worth less in terms of DM goods. Our results suggest that the second effect always dominate the third effect.

The above analyses are summarized in the following proposition.

Proposition 3. *If the labor supply is inelastic and the central bank implements monetary policy through LST only, in the steady state, we have (i) $\pi = \tau$; (ii) there is unique equilibrium for given τ; (iii) an increase of τ reduces $D_\ell^\tau(\tau)$ but has no effect on $D_u^\tau(\tau)$; (iv) if $D \geq D_u^\tau(\tau)$, then the allocation is efficient and invariant to inflation, but real wage decreases with inflation; if $D \in (D_\ell^\tau(\tau), D_u^\tau(\tau))$, then the allocation is not efficient but is invariant to inflation; (v) if $D < D_\ell^\tau(\tau)$, inflation raises $x(1)$ and reduces $x(\theta)$; and (vi) when $1+\tau = \beta$, the allocation is efficient and $D_\ell^\tau(\tau) = D_u^\tau(\tau)$.*

Figure 1 plots an example of effects of monetary policy if it is conducted by LST. We emphasize that with LST, the central bank can get the economy out of the liquidity trap equilibrium. Suppose that we are in the liquidity trap equilibrium for some given D and τ. Because $D_\ell^\tau(\tau)$ is decreasing in τ, the central bank can make sure that $D > D_\ell^\tau(\tau)$ by injecting enough money (i.e., higher τ). But moving out of the liquidity trap reduces welfare. To understand this result, first, notice the after-intervention money holding of the borrowers are $m_t + \tau M_{t-1} = M_t$, which is worth $p_t \ell$ when we are out of the liquidity trap. However, in the liquidity trap equilibrium, the existence of idle cash means a fraction of M_t is worth $p_t \ell$. In other words, the borrowers can use their after-intervention money holdings, $m_t + \tau M_{t-1}$, to purchase more DM goods than out of the liquidity trap.

At first glance, our results may appear to be different from Williamson [2012], in which LST is used to change the money supply, but it cannot move the economy out of the liquidity trap equilibrium. This is because there the quantity of bonds is assumed to be a constant fraction of the money supply. If we let money growth be higher while fixing the quantity of bonds (so that bonds account for a smaller fraction of the money supply) in Williamson [2012], then the economy also moves out of the liquidity trap.[27]

4.3 SF Only

Now we consider SF. We continue to assume that $D_t^c = \delta_t \hat{M}_t$, where δ_t represents the relative size of the central bank intervention. First, notice that the central bank lending in period t generates a profit that affects the bonds issued by the fiscal authority in period $t+1$, which, in turn, change the quantity of bonds agents that can hold to insure against liquidity shocks in period $t+2$. Following our analysis in Section 3.2.2, we have

[27] See Figure 2 of Williamson (2012).

$$q_{t+1}^{t+1}B_{t+1} = Gp_{t+1} - T_{t+1}^{b} = Gp_{t+1} - \gamma D_t^c \left(R_t - 1\right) \mathbf{1}\left(D_t^c > 0\right). \tag{47}$$

In the CM of period $t+1$, every agent will choose the same new bond position for the following FM $z_{t+2}^{t+1} = B_{t+1}$. After some algebra, we can use equation (39) to write the $x_t(\theta)$ when high-demand consumers are constrained in the FM:

$$\begin{aligned} x_t(\theta) =& \frac{\hat{M}_t}{p_t} - \frac{\gamma\delta_{t-1}}{1+\pi_t}\frac{\hat{M}_{t-1}}{p_{t-1}}\left(R_{t-1}-1\right)\mathbf{1}\left(\delta_{t-1}>0\right) + \frac{D}{\omega_t R_t} \\ &+ G\frac{R_{t-1}}{1+\pi_t} - \frac{\gamma\delta_{t-2}}{1+\pi_{t-1}}\frac{\hat{M}_{t-2}}{p_{t-2}}\left(R_{t-2}-1\right)\mathbf{1}\left(\delta_{t-2}>0\right)\frac{R_{t-1}}{1+\pi_t}, \end{aligned} \tag{48}$$

which reflects the fact that the central bank's operating profits affect both m_t (i.e., $m_t = \hat{M}_t - T_t^b$) and z_t^{t-1}. In words, more central bank profits mean more reimbursement money in the hands of the fiscal authority, so that (a) a smaller part of $\hat{M}_t$ is held by private agents; and (b) the fiscal authority issues fewer bonds. As far as we know, neither of these channels of monetary policy has been discussed before. If we are in the constrained equilibrium in period t (i.e, $R_t > 1$), we must have $\hat{M}_t/p_t = \ell/\left(1+\delta_t\right)$, which is not true in the liquidity trap equilibrium.

Conservative Intervention Consider constant δ. In principle, the two cutoffs of the borrowing limit, $D_u^\delta(\delta)$ and $D_\ell^\delta(\delta)$, are functions of δ, where the superscript, δ, indicates that SF is being used to implement monetary policy. Notice that $M_t = \hat{M}_t\left(1+\delta_t\right)$, where $m_t + T_t^b = \hat{M}_t$. We can use the condition $x(1) = u_x^{-1}(\omega R)$, the steady-state version of equation (48), the market clearing conditions and the condition of bond issuance (47) to write the two cutoffs of the borrowing limit:

$$D_u^\delta(\delta) = D_u^\tau + \bar{\omega}\left\{\frac{\delta\ell}{1+\delta} + \frac{\gamma\delta\ell}{1+\delta}\frac{\mathbf{1}(\delta>0)}{1+\delta(1-\gamma)}\frac{1-\beta^2}{\beta^2}\right\}, \tag{49}$$

$$D_\ell^\delta(\delta) = D_\ell^\tau(0) + \underline{\omega}\frac{\delta\ell}{1+\delta}, \tag{50}$$

where $\bar{\omega}$ is defined in equations (41); $\underline{\omega}$ is defined in (42) with $\pi = 0$; and D_u^τ and $D_\ell^\tau(0)$ are given by (44) and (45), respectively. Notice that if $R = 1$ and δ is constant, then $\pi^M = 0$, according to (32). We immediately have the following lemma:

Lemma 4. *With financial frictions and standing facilities only*, we have (i) $D_u^\delta(\delta)$ *and* $D_\ell^\delta(\delta)$ *are positive if* G *is low and negative if* G *is large; (ii)* $D_u^\delta(\delta)$ *and* $D_\ell^\delta(\delta)$ *are increasing in* δ *and* $D_u^\delta(\delta) \geq D_u^\tau$ *if and only if* $\delta \geq 0$.

It is useful to compare the cutoffs under LST and SF. If $\gamma = 0$, then $D_u^\delta(\delta)$ is higher than D_u^τ by $\delta\ell/(1+\delta)$. Note that $m_t = \hat{M}_t = M_t/(1+\delta)$. In words, the high-demand consumers' money stock, m_t, is enough to buy $\ell/(1+\delta)$ units of DM goods. With LST,

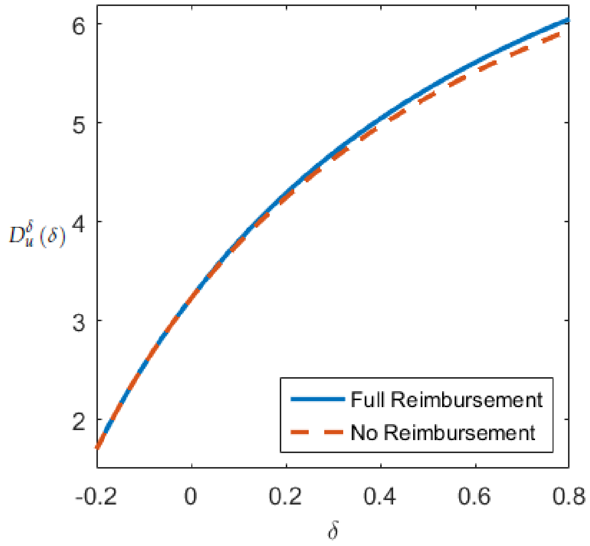

Figure 2: Effects of Central Bank Lending (δ) on $D_u^\delta(\delta)$

$m_t + \tau M_{t-1}$ is enough to buy exactly ℓ units of DM goods. SF lending redistributes liquidity away from the constrained agents. To consume the efficient (unconstrained) amount, $D_u^\delta(\delta)$ needs to be exactly $\bar{\omega}\delta\ell/(1+\delta)$ higher than D_u^τ.

If $\gamma \neq 0$, then $D_u^\delta(\delta)$ has yet another term because more central bank lending is associated with more profits and, thus, a higher reimbursement to the fiscal authority and a smaller supply of government bonds for agents to insure against shocks.[28] A higher borrowing limit is needed. Figure 2 plots examples of $D_u^\delta(\delta)$ with full reimbursement ($\gamma = 1$) and with no reimbursement ($\gamma = 0$). Notice that whenever $\delta \leq 0$, γ has no effect on $D_u^\delta(\delta)$. The comparison of $D_\ell^\delta(\delta)$ and $D_\ell^\tau(\tau)$ is similar except that, now, inflation is always zero whenever $R = 1$. According to (32), there is no profit for central banks.

After characterizing the boundary conditions, we can think about policy implications under the three regimes. First, if $D > D_u^\delta(\delta)$, both types of agents are not constrained, so we must have $\theta u_x[x(\theta)] = u_x[x(1)] = \bar{\omega} = \omega R$. An increase in δ does not affect $x(\theta)$ or $x(1)$. It affects the money growth rate and nominal interest according to (32). A higher money growth rate raises R but reduces ω, as in Section 3.

Second, if $D_u^\delta(\delta) > D > D_\ell^\delta(\delta)$, then the equilibrium conditions are steady-state versions of (37) and (38), with $p\phi = \omega$, the inflation rate given by (32), and $x(\theta)$ given by equation (39) (with z_t^{t-1}, as discussed above). We have a system of two equations and two unknowns: ω and R. An increase in δ lowers ω, R, ωR and $x(\theta)$ because it raises $D_u^\delta(\delta) - D$, which means that $x(\theta)$ is further below the efficient level. Consequently, $x(1)$ has to increase. This implies a lower ωR because $u_x[x(1)] = \omega R$.

Notice that there is a critical difference between LST and SF when agents are con-

[28] A higher δ also reduces R, according to Lemma 1. But the quantity effects dominate.

strained. With LST, the injection of money by the central bank goes to both the borrowers and the lenders. With SF, the injection of money is in the form of central bank lending and does not go to the borrowers who are already financially constrained. It is the financially unconstrained lenders who acquire such increased liquidity (by lending less). Agents with a higher marginal benefit of liquidity, thus, have relatively less liquidity in the economy. This is why δ lowers $x(\theta)$, whereas τ does not affect $x(\theta)$ at all. This is a novel channel of monetary policy. A model with LST would miss such differential effects of monetary policy on borrowers and lenders.[29]

Third, if $D < D_{\ell}^{\delta}(\delta)$, then $R = 1$, and policy changes have no marginal welfare implications. Equation (32) says that the money growth rate is zero whenever δ is constant and $R = 1$. The inflation rate π must also be zero.[30] Then, the Euler equation (40) and market clearing condition (42) pin down $\underline{\omega}$ and, thus, $x(\theta)$, which are unaffected by δ. Since $D_{\ell}^{\delta}(\delta)$ is increasing in central bank lending, δ, if the central bank increases its lending, we still have $D < D_{\ell}^{\delta}(\delta)$, and, thus, the economy cannot move out of this regime. It is like the usual notion of the liquidity trap as defined by Wikipedia: "Injections of cash .... by a central bank fail to decrease interest rates and hence make monetary policy ineffective," exactly what happens here.

Why does monetary policy affect the allocation in the constrained equilibrium, but not in the liquidity trap equilibrium? This is because central bank lending has three direct effects on borrowers in the former but none in the latter. First note that $m_t + T_t^b = \hat{M}_t$. In the constrained equilibrium, no idle cash means that $M_t = p_t \ell$. If $\gamma = 0$, then $T_t^b = 0$ and, thus, $m_t/p_t = \ell/(1+\delta_t)$. According to (39), more injection of money through SF (i.e., higher δ_t) directly diminishes the relative liquidity position of the borrowers, who can now use their after-settlement money balance, m_t, to purchase fewer DM goods. If $\gamma > 0$ and $R > 1$, then, in addition, past increases in δ_{t-1} and δ_{t-2} also affect $\hat{M}_t$ and z_{t-1}^{t-1}. Higher δ_{t-1} means higher T_t^b-i.e., the central bank reimburses more money balances to the fiscal authority, which also reduces the borrowers' relative liquidity positions. Higher δ_{t-2} reduces bond issuance in period $t-1$ and, thus, how much z_{t-1}^{t-1} private agents can carry into period t. This can also be seen in (48). However, none of these effects works in the liquidity trap equilibrium. When $R = 1$, then $T_t^b = 0$, so the second and the third effects are gone-i.e., the second and the fifth term on the RHS of (48) are zero. The first effect is also gone because we no longer have $M_t = p_t \ell$; that is, not every unit of the money balances has to be spent in the DM.

[29] In models with heterogeneous agents, LST can even generate redistributional effects that insure unlucky individuals (see, e.g., Molico 2006 and Bhattacharya et al. 2008). If these unlucky individuals are also likely to be constrained borrowers, then according to our analysis, they would not receive the newly injected money.

[30] When $R = 1$, $x_t(\theta)$ is equal to $m_t/p_t + D/\omega_t + G/(1+\pi_t)$. The problem is now that $M_t/p_t > \ell$ because of the idle cash. However, steady state implies constant π_t, and ω_t, so that $m_t/p_t = \hat{M}_t/p_t$ must also be constant for $x_t(\theta)$ to be constant. Therefore, the inflation rate must still be the same as, money growth rate.

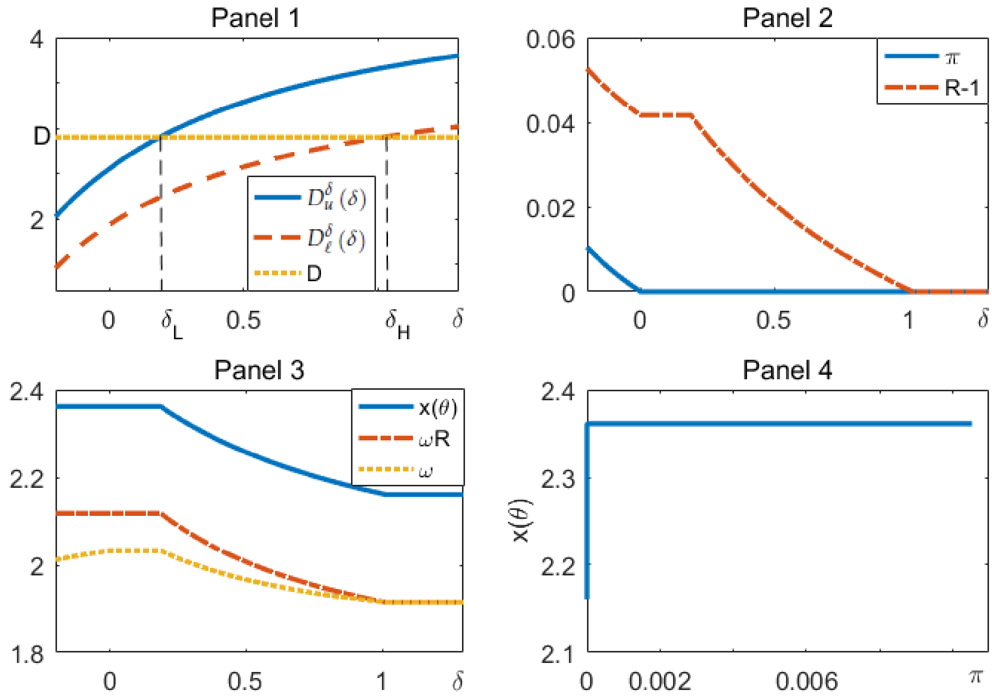

Figure 3: Effects of Central Bank Lending (δ) with Full Reimbursement ($\gamma = 1$)

Meanwhile, the general equilibrium effects do not work either. The money growth rate and inflation rate are zero if δ is constant and $R = 1$, according to (31). Hence, from (42), we know that real wage is also unaffected by δ.

The above results are summarized below and seriously question the use of LST to model monetary policy implementation: with the presence of financial frictions, you get wrong answers, if not opposite, through the lens of LST.

Proposition 4. *If the labor supply is inelastic and the central bank implements monetary policy through SF only, in the steady state with constant δ, we have (i) a unique equilibrium; (ii) if $D > D_u^\delta(\delta)$, then the allocation is efficient and δ affects the money growth rate, R, and ω but not the allocation; (iii) if $D \in \left(D_u^\delta(\delta), D_\ell^\delta(\delta)\right)$, then an increase in δ lowers ω, R, ωR and $x(\theta)$; and (iv) if $D < D_\ell^\delta(\delta)$, policy has no marginal effect on inflation and the allocation.*

What can make an economy fall into a liquidity trap equilibrium? The above analysis suggests that an exogenous decrease in D or G (i.e., so that $D < D_\ell^\delta(\delta)$) can. A decline in D could be due to a fall in the prices of the asset that are used as collateral or due to the malfunction of the financial market. A smaller G in the model can reflect a shortage of liquid assets, first used by Williamson [2012] to generate a liquidity trap equilibrium.

However, the most striking result here is that the economy can fall into the liquidity trap equilibrium endogenously due to the conduct of monetary policy. Figure 3 plots a simulated example of SF with $\gamma = 1$ (i.e., the central bank rebates all of its

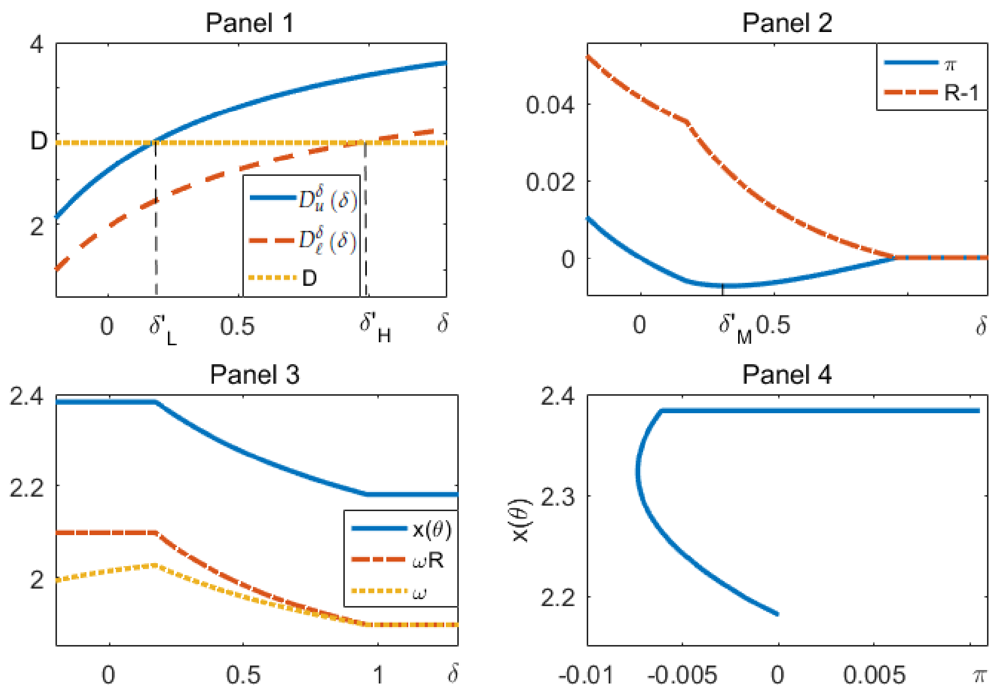

Figure 4: Effects of Central Bank Lending (δ) without Reimbursement ($\gamma = 0$)

profits). It also showcases the above analysis of the effect of SF (i.e., δ) on various endogenous variables.[31] Now, the size of the central bank's intervention in the steady state determines which equilibrium are we in: when $\delta < \delta_L$, we are in the unconstrained equilibrium; when $\delta_L < \delta < \delta_H$, we are in the constrained equilibrium; and when $\delta > \delta_H$, we are in the liquidity trap equilibrium.

In the liquidity trap, more central bank lending (i.e., higher δ) can only put the economy deeper inside the liquidity trap, and δ has no marginal effect on R or the allocation. But there is one way out, as shown by Figure 3. We need the central bank to increase borrowing in the steady state (i.e., lower δ). Even though the central bank takes liquidity away from the market, it creates a differential effects on borrowers and lenders: lenders are happy to lend to the central bank because they have extra liquidity (i.e., idle cash), whereas constrained borrowers would not lend to the central bank because they are short of liquidity themselves. The relative liquidity positions of the borrowers improve, and, thus, they are less affected by the borrowing constraint (i.e., $D_\ell^\delta(\delta)$ would decrease). If δ is low enough so that $D > D_\ell^\delta(\delta)$, then we are out of the liquidity trap. Moreover, moving out of the liquidity trap is welfare-improving. These results are not just novel but also completely opposite to those under LST: there, the

[31] Notice that because $\gamma = 1$, the central bank rebates all of its profits. Any positive δ implies a zero inflation rate. When $0 < \delta < \delta_L$, a change of δ has no effect on R because the inflation rate is constant, and we are in the unconstrained equilibrium. Thus, ω is also not affected (because $\omega R = \bar{\omega}$ is constant in the unconstrained equilibrium).

central bank can affect inflation in the liquidity trap (when $R = 1$), and by injecting more money, gets the economy out of the liquidity trap, which lowers welfare.

There is another novel result. With LST, there is a one-to-one mapping between policy action, τ, and the money growth rate. With SF and $\gamma = 1$, any positive δ is associated with zero inflation, which know from Section 3, equation (32). However, suppose that $\delta > 0$ and $D_u^\delta(\delta) > D > D_\ell^\delta(\delta)$. Then, a change in δ can affect real allocations even if it does not change the money growth rate and inflation. This is shown in Figure 3 when $\delta_H > \delta > \delta_L$. The money growth rate is not a sufficient statistic for monetary policy. There are two important implications. First, the central bank can improve welfare without changing inflation. Second, monetary policy rules based on the money growth rate can be ineffective, whereas rules based on the nominal interest rate are effective.

If $\gamma < 1$, there are also several interesting results. Figure 4 plots a simulated example in which there is a U-shaped relationship between δ and inflation/money growth rate (Panel 2). From equation (32), we can see that there are two opposing effects of δ that are responsible for such a U-shaped relationship: (a) given R, an increase in δ lowers inflation if $\gamma < 1$ and $\delta > 0$; and (b) a higher δ also lowers R, which raises π. The second effect is always dominated in the unconstrained equilibrium but can be dominating in the constrained equilibrium. The U-shaped relationship between δ and inflation then implies a backward-bending relationship between welfare and inflation (note that the higher $x(\theta)$, the higher the welfare). When $\delta \in \left(\delta_L^{'}, \delta_M\right)$, welfare appears to be increasing in inflation/money growth rate; when $\delta \left(\delta_M, \delta_H^{'}\right)$, welfare appears to be decreasing in inflation/money growth rate. This again suggests that the central bank cannot focus only on the money growth rate. If we let labor be elastic, then DM output is also backward-bending with inflation when $\gamma < 1$. If $\gamma = 1$, there will be a range of δ with different DM outputs but the same inflation rate, and another range of δ in which DM outputs appear to be decreasing in inflation.

In Williamson [2012] and in our analysis with LST, the central bank can always marginally affect the allocation in the liquidity trap equilibrium because it can use LST to change the money supply. But, here, once we are explicit about how the central bank changes the money supply, we show that changing δ cannot marginally affect inflation because $\pi = 0$ whenever $R = 1$.

Aggressive Intervention Next, consider changing δ_t over time. Suppose that the central bank wants to implement a steady-state equilibrium with constant π^M, R, and ω. Again, the money growth rate is given by condition (31). Suppose that we are in the constrained equilibrium, so that $\hat{M}_t/p_t = \ell/(1+\delta_t)$. If $\gamma = 0$, then, according to equation (48), a constant $x_t(\theta)$ implies a unique and constant δ_t. The aggressive intervention is reduced to a conservative intervention. Then, suppose that $\gamma = 1$, so

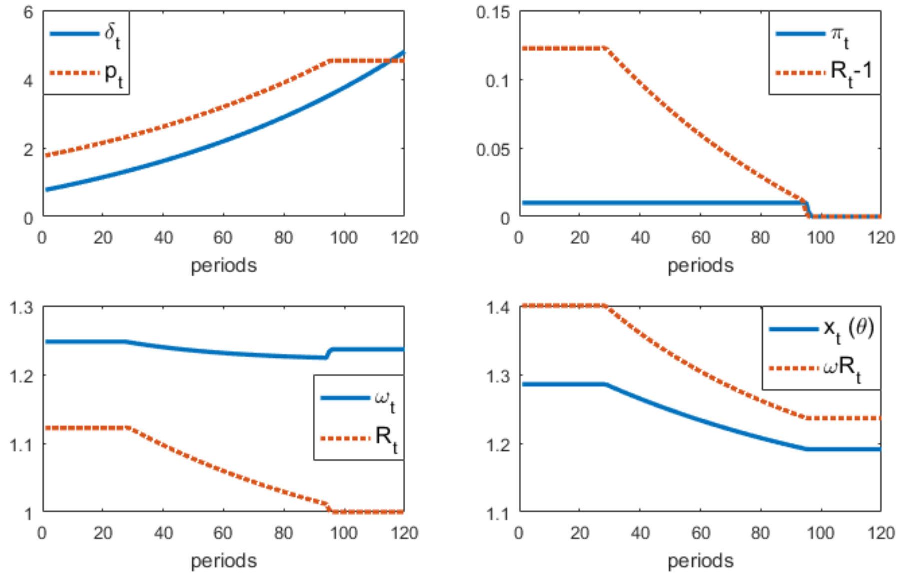

Figure 5: Aggressive Lending with Positive Inflation Target and Financial Frictions

that $(1+\delta_t)/(1+\delta_{t-1})$ is equal to $1+\pi^M$. Now, equation (48) implies that $x_t(\theta)$ cannot be constant. In general, the central bank cannot use the aggressive intervention to implement a constant inflation rate in the steady-state constrained equilibrium.

Moreover, equation (48) also suggests that the upper limit of the borrowing constraint (denote it by $D^{\delta}_{u,t}(\delta_t)$) can change over time because it is exactly the value of D that makes $x_t(\theta) = u_x^{-1}(\bar{\omega})$ when $\omega_t R_t = \bar{\omega}$. If δ_t increases over time, then $D^{\delta}_{u,t}(\delta_t)$ would also increase over time. The intuition is that as the central bank increases its lending over time, borrowers can afford less goods in the DM using their own money balances (i.e., $m_t/p_t = \ell/(1+\delta_t) - T^b_t/p_t$), making it harder for them to be unconstrained (tighter borrowing constraint). The effect of aggressive intervention on $D^{\delta}_{\ell,t}(\delta_t)$ is similar. Therefore, even if, initially, agents are not constrained (i.e., $D > D^{\delta}_{u,t}(\delta_t)$), increasing δ_t would eventually make sure that $D < D^{\delta}_{u,t}(\delta_t)$. Since $D^{\delta}_{\ell,t}(\delta_t)$ is also increasing in δ_t, the economy would endogenously go into the liquidity trap.

Figure 5 plots such an example with a finite D and $(1+\delta_t)/(1+\delta_{t-1}) = 1\%$, which can be used to implement 1% of inflation rate if there were no financial frictions.[32] Starting from Period 28, the borrowing constraint of the high-demand consumers becomes binding, and nominal interest starts to fall below that implied by the Fisher equation until it sharply drops to zero in Period 96. The intuition is that when borrowing constraint is binding, further injection of the liquidity depress the nominal interest rate more, i.e., a lower nominal interest rate is required to clear the loan mar-

[32]For simplicity, we let $\gamma = 1$ and assume that the central bank's profit is used to reduce the CM lump-sum taxes by the government, i.e., central bank profits do not affect the bond issuance.

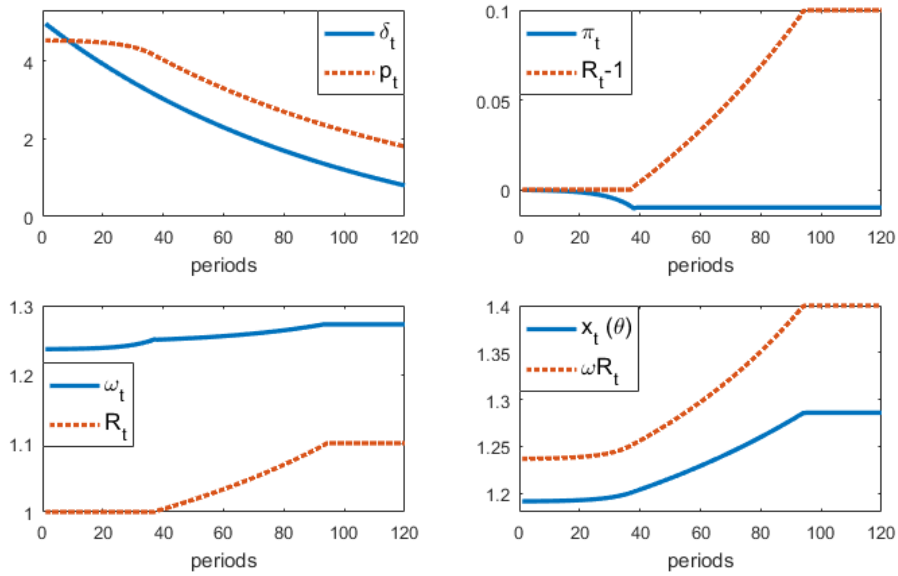

Figure 6: Aggressive Lending with Negative Inflation Target and Financial Frictions

ket.[33] Welfare, as indicated by $x_t(\theta)$, decreases as well, because borrowers' relative liquidity positions are weakened. Real wage decreases but bounces up in Period 96 due to a sharp drop in R. Inflation rate becomes zero in Period 97; and the economy settles in the liquidity trap equilibrium despite the central bank's further injection of money.

Interestingly, the central bank simply wants to implement a positive and constant inflation rate. The problem is that it ignores how the aggressive option interact with financial frictions. With a conservative intervention, the size of the intervention determines which equilibrium are we in. But, here, the aggressive intervention can lead the economy to dynamically experience the three types of equilibrium.

On the top of that, the dynamic process can also be reversed. Figure 6 plots such an example with a finite D and $(1+\delta_t)/(1+\delta_{t-1}) = -1\%$. The economy moves out of the liquidity trap in Period 38, but, interestingly, nominal price level starts to change before that. In the previous example, prices are stuck as soon as we enter the liquidity trap, because agents expect the economy to be stuck in the liquidity trap equilibrium given the prevailing monetary policy. In the current example, prices change even though $R_t = 1$ because of the expectation of escaping the liquidity trap. Also, note that $D^{\delta}_{u,t}(\delta_t)$ and $D^{\delta}_{\ell,t}(\delta_t)$ would decrease over time, and, thus, we achieve the unconstrained equilibrium in the long run.

This "escape" strategy is not only novel but also very different from what a model of LST predicts. Therefore, a smaller steady-state money growth rate improves wel-

[33] The borrowers can at most borrow $D/R\phi$ units of money. When the supply of fundings increases in the loan market, R would decrease to balance the supply and demand.

fare, though we can only get out of the liquidity trap by a much higher steady-state money growth rate. Here, the aggressive strategy would take some time for the central bank to escape the liquidity trap: for a while, economy appears to be stuck in the liquidity trap; but, over time, there is less and less idle cash in the DM; and the economy gets into the constrained equilibrium and the unconstrained equilibrium eventually.

4.4 OMO Only

As in Section 3, the central bank trades either old bonds or new bonds, and we still use $\mu_{ot} = q_t^{t-1} Z_t^{t-1} / \hat{M}_t$ and $\mu_{nt} = q_t^t Z_t^t / \hat{M}_t$ to denote the relative size of intervention. If the central bank trades only old bonds, then OMO is the same as with SF. We can simply replace the δ_t in our analysis in Section 4.3 with μ_{ot}. Of course, μ_{ot} and μ_{nt} cannot be negative unless the central bank can issue its own bonds.

Proposition 5. *If the labor supply is inelastic and the central bank implements monetary policy through OMO of old bonds only, then Lemma 4 and Proposition 4 hold if we replace δ with μ_o.*

We will focus on the interesting case in which the central bank trades new bonds. This is also more realistic. We are not saying that central banks only purchase only newly issued bonds in the real world, but this case captures the fact that if the central bank holds more of these bonds, then private agents can hold fewer across periods – another way of saying that the bonds that central banks purchase generally would not mature immediately. Specifically, the fiscal authority issues bonds B_{t+1}, similar to equation (47), in which we will replace D_t^c with $q_t^t Z_t^t$. However, in the CM of period $t+1$, private bond holding z_{t+2}^{t+1} is now $B_{t+1} - Z_{t+1}^{t+1}$ rather than B_{t+1}, as in our analysis of SF.

Conservative Intervention Consider a constant relative size of intervention, μ_n. The two cutoffs of the borrowing constraint are now

$$D_u^{\mu_n}(\mu_n) = D_u^{\delta}(\mu_n) + \bar{\omega}\frac{1}{\beta}\frac{\ell\mu_n}{1+\mu_n}, \tag{51}$$

$$D_\ell^{\mu_n}(\mu_n) = D_\ell^{\delta}(\mu_n) + \underline{\omega}\frac{\ell\mu_n}{1+\mu_n}, \tag{52}$$

where $\bar{\omega}$ is defined in equation (41) and $\underline{\omega}$ is defined in equation (42) with $\pi = 0$. We immediately have the following lemma:

Lemma 5. *With financial frictions and standing facilities only*, we have (i) $D_u^{\mu_n}(\mu_n)$ *and* $D_\ell^{\mu_n}(\mu_n)$ *are positive if G is low and negative if G is large; (ii) $D_u^{\mu_n}(\mu_n)$ and $D_\ell^{\mu_n}(\mu_n)$ are increasing in μ_n; and (iii)$D_u^{\mu_n}(\mu_n) \geq D_u^{\delta}(\mu_n) \geq D_u^{\tau}(\tau)$ if and only if $\mu_n \geq 0$.*

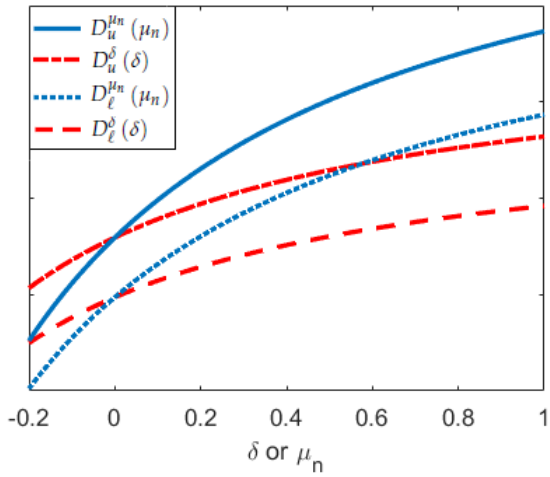

Figure 7: Cutoffs of Borrowing Limit under SF and OMO

The difference between $D_u^{\mu_n}(\mu_n)$ and $D_u^{\delta}(\mu_n)$ and the difference between $D_\ell^{\mu_n}(\mu_n)$ and $D_\ell^{\delta}(\mu_n)$ are due to the fact that if the central bank uses OMO, then private agents can hold fewer government bonds to insure against their idiosyncratic shocks. We call it the "*crowding-out*" effect of OMO. Given the same relative intervention size (i.e., $\mu_n = \delta$), if the central bank does not issue its own bonds (i.e., $\mu_n > 0$), then OMO makes the borrowing constraints tighter compared to SF. Of course, this relationship is reversed if the central bank issues its own bonds to implement some $\mu_n < 0$.

Next, consider the policy implications under the three regimes. First, if $D > D_u^{\mu_n}(\mu_n)$, both types of agents are not constrained; the allocation is efficient; and an increase in μ_n does not affect $x(\theta)$ or $x(1)$. It affects the money growth rate according to (35). A higher money growth rate raises R but reduces ω, as in Section 3.

Second, if $D_u^{\mu_n}(\mu_n) > D > D_\ell^{\mu_n}(\mu_n)$, then the equilibrium conditions are steady-state versions of (37) and (38), with $p\phi = \omega$, the inflation rate, given by (35), and $x(\theta)$ given by equation (39) (with z_t^{t-1} discussed above). We have a system of two equations and two unknowns: ω and R. An increase in μ_n lowers ω, R, ωR and $x(\theta)$ because it raises $D_u^{\mu_n}(\mu_n) - D$, which means $x(\theta)$ that is further below the efficient level. Such an effect is stronger than SF because of the illiquid effect of OMO (i.e., the extra term in equation 51).

Third, if $D < D_\ell^{\mu_n}(\mu_n)$, then $R = 1$ and policy changes have no marginal welfare implications. First, equation (35) says that the money growth rate is zero if $R = 1$. Now, the consumptions of high-demand consumers are given by

$$x_t(\theta) = \frac{m_t}{p_t} + \frac{D}{\omega_t} + G/(1+\pi_t) - \frac{m_{t-1}}{p_{t-1}}\mu_n, \tag{53}$$

which is different from that in Footnote 23. The last term, $m_{t-1}\mu_n/p_{t-1}$, is absent in

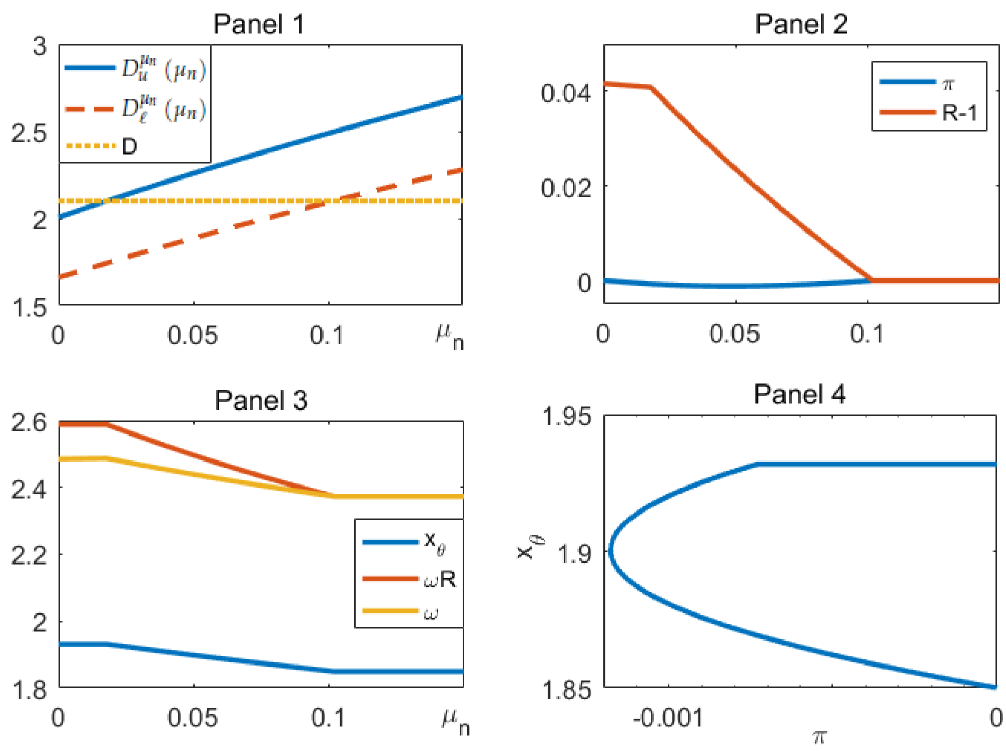

Figure 8: Effects of Central Bank OMO (μ_n) with No Reimbursement

Footnote 23, which reflects the fact that more OMO reduces the amount of the bonds that private agents can hold. In a liquidity trap, we cannot directly pin down m_t/p_t because $M_t/p_t > \ell$ due to the idle cash that agents carry in the DM. But the steady state implies constant π_t and ω_t, so that $m_t/p_t = m_{t-1}/p_{t-1}$ must also be constant for $x_t(\theta)$ to be constant. Therefore, the inflation rate must still be the same as the money growth rate, which is zero in this case. Then, the Euler equation (40) and market clearing condition (42) pin down $\underline{\omega}$ and, thus, $x(\theta)$, all of which are unaffected by μ_n. Since $D_\ell^{\mu_n}(\mu_n)$ is increasing in the central bank intervention, μ_n, we cannot move out of this regime if the central bank purchases more bonds in the steady state. It is again like the usual notion of the liquidity trap. Figure 3 plots an example of OMO with $\gamma = 1$.

There is another difference between OMO and SF. The steady-state version of equation (53) implies that $m/p - \mu_n m/p$ is invariant to μ_n. It means that more "injection" of money (e.g., a higher μ_n) in a liquidity trap causes the real money balances that agents carry across periods, $m/p = \hat{M}/p$, to increase. Notice that under SF, $\hat{M}/p$ is invariant to δ in the liquidity trap.[34] However, OMO is still ineffective in a liquidity trap because such a change of real money balances and the crowding-out effect of OMO exactly offset each other.

[34]Under both frameworks, the real money balances in the DM,M/p, increase with greater intervention (i.e., a higher δ or μ_n). But they increase more under the OMO.

Similar to our analysis of SF, if $\gamma < 1$, then we might have a U-shaped relationship between μ_n and the money growth rate if $D \in \left(D_u^{\mu_n}(\mu_n), D_\ell^{\mu_n}(\mu_n)\right)$ and $\mu_n > 0$. The above results are summarized below.

Proposition 6. *If the labor supply is inelastic and the central bank implements monetary policy through OMO of new bonds only, in the steady state with constant δ, we have (i) a unique equilibrium; (ii) if $D > D_u^{\mu_n}(\mu_n)$, then the allocation is efficient and μ_n affects the money growth rate, R, and ω but not the allocation; (iii) if $D \in \left(D_u^{\mu_n}(\mu_n), D_\ell^{\mu_n}(\mu_n)\right)$, then an increase in μ_n lowers ω, R, ωR and $x(\theta)$; and (iv) if $D < D_\ell^{\delta}(\delta)$, policy has no marginal effect on inflation and the allocation.*

Aggressive Intervention Next, consider the case of changing μ_{nt}. Suppose that, in period t, we are in the constrained equilibrium (i.e., binding borrowing constraint and no idle cash), we can rewrite the expression of $x_t(\theta)$ (39) as

$$x_t(\theta) = \frac{\ell}{1+\mu_t} + \frac{D}{\omega_t R_t} + G\frac{R_{t-1}}{1+\pi_t} - \gamma\frac{\ell\mu_{nt-2}}{1+\mu_{nt-2}}\frac{R_{t-2}-1}{1+\pi_{t-1}}\frac{R_{t-1}}{1+\pi_t} - \frac{R_{t-1}}{1+\pi_t}\frac{\ell\mu_{nt-1}}{1+\mu_{nt-1}}. \tag{54}$$

There are two observations. First, by comparing the above equation and equation (48), we know that if $\mu_t = \delta_t$ for every t, then $x_t(\theta)$ is smaller under OMO than under SF. Second, similar to our analysis of SF, there is no steady state if the central bank uses the aggressive option for implementing some constant inflation rate.

Expanding μ_{nt} over time would send the economy into the liquidity trap, and OMO would be more "effective" than SF in this regard due to the crowding-out effect. Of course, this means that if the central bank shrinks μ_{nt} over time (i.e., by holding fewer government bonds over time or issue more of its own bonds), we would also get out of the liquidity trap faster than with SF. This is particularly interesting given the discussion of exit strategies of the central banks in the post crisis-era.

5 Concluding Remarks

This paper has both methodological and practical implications. In terms of methodology, our comparative approach showcases what we call the "FM principle" about modeling the implementation of monetary policy: if the financial market is imperfect, then it is important to model the specifics of SF and OMO; otherwise, LST suffices as a modeling device. The FM principle is useful because in many existing models, especially ones with homogeneous agents, the requirement of perfect financial market is trivially true. Other models, which emphasize the redistributional effects of LST, might be suspect according to our analysis.

We also show a novel rationale favoring a nominal interest rate over the money supply as an intermediate target of central banks – if the financial market is imperfect, different interventions of SF or OMO alone can lead to different allocations, even though the steady-state money growth rate is endogenously the same.[35] Previous arguments against targeting the money supply concern the difficulty of measuring and the prevalence of shocks to the demand for money in the short run. What we show is different: even if a model focuses on the steady-state implications of monetary policy, it still should not treat the money supply as a target of the central bank if there are financial frictions in the model. Our results also imply that the apparent long-run relationship between the money growth rate and other real variables might be superficial. This can be important for empirical studies of the long-run effects of inflation.

On the practical side, our results regarding liquidity traps have many policy implications. By comparing LST and SF/OMO, we point out an important reason that money injection, in the form of more central bank lending or purchase of bonds, does not work in the liquidity trap equilibrium: the newly injected money does not go to the borrowers because they are already financially constrained. Monetary policy still affects the allocation in the constrained equilibrium because of the general equilibrium effects that are absent in the liquidity trap equilibrium (nominal interest rate changes, as well as how much goods the borrowers can use their own money to purchase). In reality, the rates faced by borrowers are usually higher than zero, even if the policy rate is zero. But a zero policy rate implies that the banks are satiated with liquidity and unable to lend more to the borrowers due to their financial constraints.[36] Hence, the above intuition carries through: if the central bank injects more money through SF or OMO, then the newly injected money does not reach the borrowers and, thus, become idle reserves kept by the banks.

We also show how to escape the liquidity trap. If the central bank uses SF or OMO conservatively, then significantly reducing its lending or purchase of bonds can eliminate the liquidity trap and improve welfare. The central bank can also consider shrinking its lending or purchase of bonds over time. However, it takes time to escape the liquidity trap, so, at first, it may appear that the policy is not working.

Another novel element of the paper is to model the aggressive way of applying SF and OMO. It is more like the actual conduct of monetary policy: that is, to achieve a positive inflation target, a central bank lends more and more to the market. However, we find an inherent problem in implementing monetary policy this way: with the presence of financial frictions, the aggressive implementation of monetary policy

[35]Williamson [2012] and Rocheteau et al. [2015] also emphasize the effects of OMO, given the same money growth rate. Their results are different because they assume in addition to OMO, the central bank can also uses LST to maintain the same money growth rate.

[36]A constrained individual borrower may face positive nominal loan rates for many reasons, such as his or her credit condition (i.e., the riskiness of the loan), and the market power of the banks.

inevitably leads to the liquidity trap endogenously over time. To avoid such a problem, the central bank should adopt the counterintuitive conservative way of obtaining a positive inflation target: by continuing to borrow and using its interest payment to expand the money supply over time.

Future study can consider how to incorporate aggregate uncertainty and various kinds of short-run frictions and rigidities. It could be also useful to explicitly model banks.

Appendix

Proof of Lemma 5:

Due to (24), the nominal interest rate of the short-term financial market is the same as the implied nominal yield of the liquid government bond. If $R < (1+\pi)/\beta$, then there appears to be a liquidity premium on the bond in the FM. Then, consider the new bonds in the CM in period t. Their nominal price is $\hat{q}_t^t = E_t\left(1/R_{t+1}\right)$. Their nominal price in the CM of period $t+1$ is 1 because, according to (2), that is how much a unit of new bonds can affect m_{t+2}. So, the gross nominal interest rate implied by the bond price is $1/\hat{q}_t^t$, which is equal to R in the steady state. The implied real interest rate of the bonds (across two CMs) is then $R/(1+\pi) < r$. Usually the real interest rate across two CM's should be exactly r due to our quasilinear utility assumption. This means that the price of bonds in the CM also contains a liquidity premium.

LST with financial friction

$$\ell - G = \alpha\left[\ell + G\frac{R}{1+\pi} + \frac{D}{\omega R}\right] + (1-\alpha)\, u_x^{-1}\left[\omega R\right],$$
$$\frac{1+\pi}{R\beta}\omega R = \alpha\theta u_x\left[\ell + G\frac{R}{1+\pi} + \frac{D}{\omega R}\right] + (1-\alpha)\,\omega R.$$

References

Fernando Alvarez, Andrew Atkeson, and Chris Edmond. Sluggish responses of prices and in flation to monetary shocks in an inventory model of money demand. *Quarterly Journal of Economics*, 124(3):911–967, 2009.

David Andolfatto. Essential interest-bearing money. *Journal of Economic Theory*, 145(4): 1495–1507, 2007.

David Andolfatto. Incentive-feasible deflation. *Journal of Monetary Economics*, 60(60): 383–390, 2013.

Adrien Auclert. Monetary policy and the redistribution channel. 2016.

Jess Benhabib, Stephanie Schmitt-Grohé, and Martin Uribe. Avoiding liquidity traps. *Journal of Political Economy*, 110(3):535–563, 2002.

Aleksander Berentsen and Cyril Monnet. Monetary policy in a channel system. *Journal of Monetary Economics*, 55:1067šC1080, 2008.

Aleksander Berentsen, Gabriele Camera, and Christopher Waller. Money, credity and banking. *Journal of Economic Theory*, 135(1):171–195, July 2007.

Aleksander Berentsen, Alessandro Marchesianic, and Christopher J. Waller. Floor systems for implementing monetary policy: Some unpleasant fiscal arithmetic. *ReviewofEconomicDynamics*, 17:523šC542, 2014.

Ben S. Bernanke and Frederic S. Mishkin. Inflation targeting: A new framework for monetary policy? *Journal of Economic Perspectives*, 11(2):97–116, Spring 1997.

Ben S Bernanke, Mark Gertler, and Simon Gilchrist. The financial accelerator in a quantitative business cycle framework. *Handbook of macroeconomics*, 1:1341–1393, 1999.

Truman Bewley. A difficulty with the optimum quantity of money. *Econometrica*, 51 (5):1485–1504, 1983.

Joydeep Bhattacharya, Joseph Haslag, Antoine Martin, and Rajesh Singh. Who is afraid of the friedman rule? *Economic Inquiry*, 46(2):113–130, 2008.

John Bryant and Neil Wallace. Open-market operations in a model of regulated, insured intermediaries. *Journal of Political Economy*, 88(1):146–173, 1980a.

John Bryant and Neil Wallace. Open-market operations in a model of regulated, insured intermediaries. *Journal of Political Economy*, 88(1):146–173, 1980b.

Lawrence J. Christiano, Martin Eichenbaum, and Charles L. Evans. Nominal rigidities and the dynamic effects of a shock to monetary policy. *Journal of Political Economy*, 113(1):1–45, 2005.

Vasco Cúrdia and Michael Woodford. Conventional and unconventional monetary policy. *FEDERAL RESERVE BANK OF ST. LOUIS REVIEW*, 92(4):229–64, 2010.

Gauti B Eggertsson and Paul Krugman. Debt, deleveraging, and the liquidity trap: A fisher-minsky-koo approach. *The Quarterly Journal of Economics*, 127(3):1469–1513, 2012.

Jordi Gali. *Monetary policy, inflation and the business cycle: an introduction to the New Keynesian Framework*. Princeton University Press, 2008.

Mark Gertler and Peter Karadi. A model of unconventional monetary policy. *Journal of monetary Economics*, 58(1):17–34, 2011.

Jean-Michel Grandmont and Yves Younes. On the efficiency of a monetary equilibrium. *Review of Economic Studies*, 40:149–165, 1973.

Sanford Grossman and Laurence Weiss. A transactions-based model of the monetary transmission mechanism. *The American Economic Review*, 73(5):871–880, 1983.

Chao He, Randall Wright, and Yu Zhu. Housing and liquidity. *Review of Economic Dynamics*, 18(3):435–455, 2015.

Greg Kaplan, Benjamin Moll, and Giovanni L. Violante. Monetary policy according to hank. 2016.

Nobuhiro Kiyotaki and John Moore. Liquidity, business cycles, and monetary policy. Technical report, National Bureau of Economic Research, 2012.

Nobuhiro Kiyotaki and Randall Wright. On money as a medium of exchange. *Journal of Political economy*, 97(4):927–954, 1989.

Meir Kohn. In defense of the finance constraint. *Economic Inquiry*, 19(2):177–95, 1981.

Paul R Krugman, Kathryn M Dominquez, and Kenneth Rogoff. It's baaack: Japan's slump and the return of the liquidity trap. *Brookings Papers on Economic Activity*, 1998(2):137–205, 1998.

Ricardo Lagos and Randall Wright. A unified framework for monetary theory and policy analysis. *Journal of political Economy*, 113(3):463–484, 2005.

Robert Lucas. Equilibrium in a pure currency economy. *in J.H. Kareken and N. Wallace, eds., Models of monetary economies. Minnieaplis: Federal Reserve Bank of Mineapolis.*, 1980.

Robert E. Lucas and Nancy L. Stokey. Money and interest in a cash-in-advance economy. *Econometrica*, 55(3):491–513, 1987.

Antoine Martin and Cyril Monnet. Monetary policy implementation frameworks: A comparative analysis. *Macroeconomic Dynamics*, 15:145–189, 2011.

Miguel Molico. The distribution of money and prices in search equilibrium. *International Economic Review*, 47(3):701–722, 2006.

Joseph M. Ostroy and Ross M. Starr. Chapter 1 the transactions role of money. *Handbook of Monetary Economics*, 1(21):3–62, 1990.

William Poole. Optimal choice of monetary policy instruments in a simple stochastic macro model*. *The Quarterly Journal of Economics*, 84(2):197–216, 1970. doi: 10.2307/1883009.

Guillaume Rocheteau and Randall Wright. Money in search equilibrium, in competitive equilibrium, and in competitive search equilibrium. *Econometrica*, 73(1):175–202, Jan. 2005.

Guillaume Rocheteau, Randall Wright, and Sylvia Xiaolin Xiao. Open market operations. 2015.

J. Rotemberg. A monetary equilibrium model with transactions costs. *Journal of Political Economy*, 92:40–58, 1984.

Shouyong Shi. A divisible search model of fiat money. *Econometrica: Journal of the Econometric Society*, pages 75–102, 1997.

Miguel Sidrauski. Rational choice and patterns of growth in a monetary economy. *The American Economic Review, Papers and Proceedings*, 57(2):534–544, 1967.

Lars EO Svensson. The zero bound in an open economy: A foolproof way of escaping from a liquidity trap. Technical report, National Bureau of Economic Research, 2000.

James Tobin. Money and economic growth. *Econometrica*, 33:671–684, 1965.

Carl E Walsh. *Monetary theory and policy*. 2017.

Ivan Werning. Managing a liquidity trap: Monetary and fiscal policy. Technical report, National Bureau of Economic Research, 2011.

Stephen Williamson, Randall Wright, et al. New monetarist economics: Models. *Handbook of Monetary Economics*, 3(11):25–96, 2011.

Stephen D. Williamson. Monetary policy and distribution. *Journal of Monetary Economics*, 55(6):1038–1053, 2008.

Stephen D Williamson. Liquidity, monetary policy, and the financial crisis: A new monetarist approach. *The American Economic Review*, 102(6):2570–2605, 2012.

Stephen D Williamson and Randall Wright. New monetarist economics: Methods. *Federal Reserve Bank of St Louis Review*, 2010.

Michael Woodford. *Interest and prices : foundations of a theory of monetary policy*. Princeton University Press, 2003.